Complexity of Shifted-Lacunary Polynomial Interpolation

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Outline

Complexity of Shifted-Lacunary Polynomial Interpolation

General Problem

Determining a function from its values.

Goals

- Find the simplest possible formula.
- Don’t take too long.

Necessities

- What type of function? (output type)
- How big can it be? (output size)

Complexity of Shifted-Lacunary Polynomial Interpolation

Example

$$f(x) = (x-3)^{107} - 485(x-3)^{54}$$

Suppose we can evaluate $f(x)$ at any chosen point.

- Can we find a formula for $f(x)$?

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Example

$$f(x) = (x-3)^{107} - 485(x-3)^{54}$$

Suppose we can evaluate $f(x)$ at any chosen point.

- Can we find a simple formula for $f(x)$ in a reasonable amount of time?

Complexity of Shifted-Lacunary Polynomial Interpolation

Dense Methods

Definition (Dense Representation)

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n,$$

where $n = \deg(f)$ and $a_0, a_1, \ldots, a_n \in \mathsf{R}$

- Studied by Newton (1711), Waring (1779), ...
- Highly efficient implementations available

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where $n = \deg(f)$ and $a_0, a_1, \ldots, a_n \in \mathsf{R}$

Example

For $f(x) = (x-3)^{107} - 485(x-3)^{54}$, we will have

$$f(x) = x^{107} - 321x^{106} + 51039x^{105} - 5359095x^{104} + \cdots$$
$$+ 402009927496590798548375851523117926743035901448193 73x$$
$$- 1127130637840908780976768693419860197828458989848152$$

This is way too big! (twice exponential in the desired size)

Complexity of Shifted-Lacunary Polynomial Interpolation

Confusing Nomenclature

$$\text{Lacunary} \subseteq \text{Supersparse} \neq \text{Sparse}$$

- Default representation in Maple, Mathematica, etc.
- Some things are hard (Plaisted 1977, 1984)
- Some things aren't: Interpolation, finding low-degree factors
- Some things are unknown!

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Sparse Methods

Definition (Lacunary Representation)

$$f(x) = b_1 x^{d_1} + b_2 x^{d_2} + \cdots + b_s x^{d_s},$$

where $d_1 < d_2 < \cdots < d_s = n$ and $b_1, \ldots, b_s \in \mathsf{R} \setminus \{0\}$

- Baron de Prony (1795), Ben-Or & Tiwari (1988), Kaltofen, Lakshman, Wiley, Lee, Lobo, ...
- Need to choose evaluation points
- R must have a high-order element and a fast logarithm.

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Sparse Methods

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where $d_1 < d_2 < \cdots < d_s = n$ and $b_1, \ldots, b_s \in \mathsf{R} \setminus \{0\}$

Example

If $f(x) = (x-3)^{107} - 485(x-3)^{54}$,
this helps iff we know the sparsest shift 3,
since $f(x+3) = x^{107} - 485x^{54}$ is 2-sparse.

Complexity of Shifted-Lacunary Polynomial Interpolation

Definition (Shifted-Lacunary Representation)

$$f(x) = c_1(x-\alpha)^{e_1} + c_2(x-\alpha)^{e_2} + \cdots + c_t(x-\alpha)^{e_t},$$

where $e_1 < \cdots < e_t = n$ and t is minimal for any α

- This is our problem.
- Can be reduced to finding the sparsest shift α.
- We restrict the domain to $\mathbb{Q}[x]$.
- No previous polynomial-time algorithm known.

Complexity of Shifted-Lacunary Polynomial Interpolation

We give an algorithm with
output-sensitive polynomial-time complexity,
specifically, *bit complexity* polynomial in:

- Number of nonzero terms t
- Logarithm of the degree n
- Size of the coefficients $c_1, \ldots, c_t$
- Size of the sparsest shift α

Black box calls are assumed to have constant cost.

Computing the Sparsest Shift

- Borodin & Tiwari (1991)
  Compute sparsest shift from evaluation points (open)
- Grigoriev & Karpinski (1993)
  Compute sparsest shift from a black-box function.
  State need for complexity *not* polynomial in n
- Lakshman & Saunders (1996)
  Compute sparsest shift from dense representation
- Giesbrecht, Kaltofen, Lee (2003)
  Current best results (deterministic & probabilistic)

Uniqueness and Rationality of Sparsest Shift

Theorem (Lakshman & Saunders (1996))

If the degree is at least twice the sparsity,
then the sparsest shift is unique and rational.

Example

$$f(x) = (x-3)^{107} - 485(x-3)^{54}$$

$\Rightarrow$ 3 is the only shift with ≤ 54 terms

Condition not satisfied means polynomial is dense.

Black Box Model

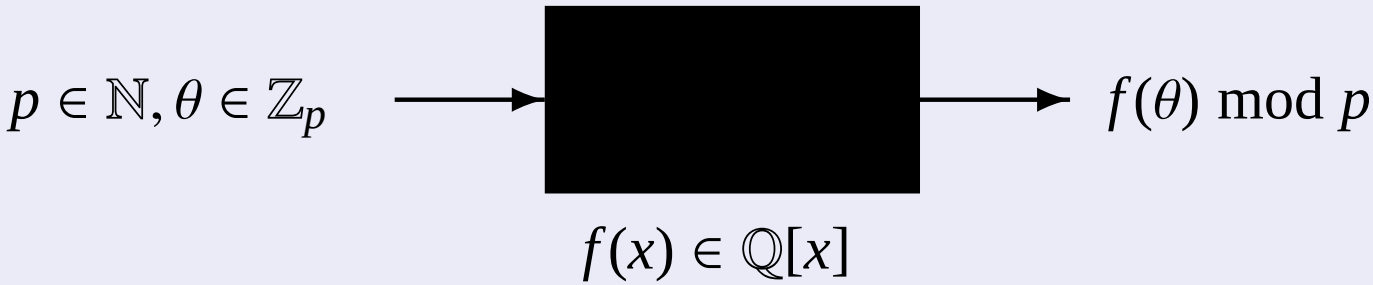
Arbitrary evaluations will usually be very large:

Example

$$f(x) = (x-3)^{107} - 485(x-3)^{54}$$
$$f(1) = -162259276829222100374855109050368$$

To control evaluation size, use modular arithmetic:

The “Modular Black-Box”



Modular-Reduced Polynomial

Definition

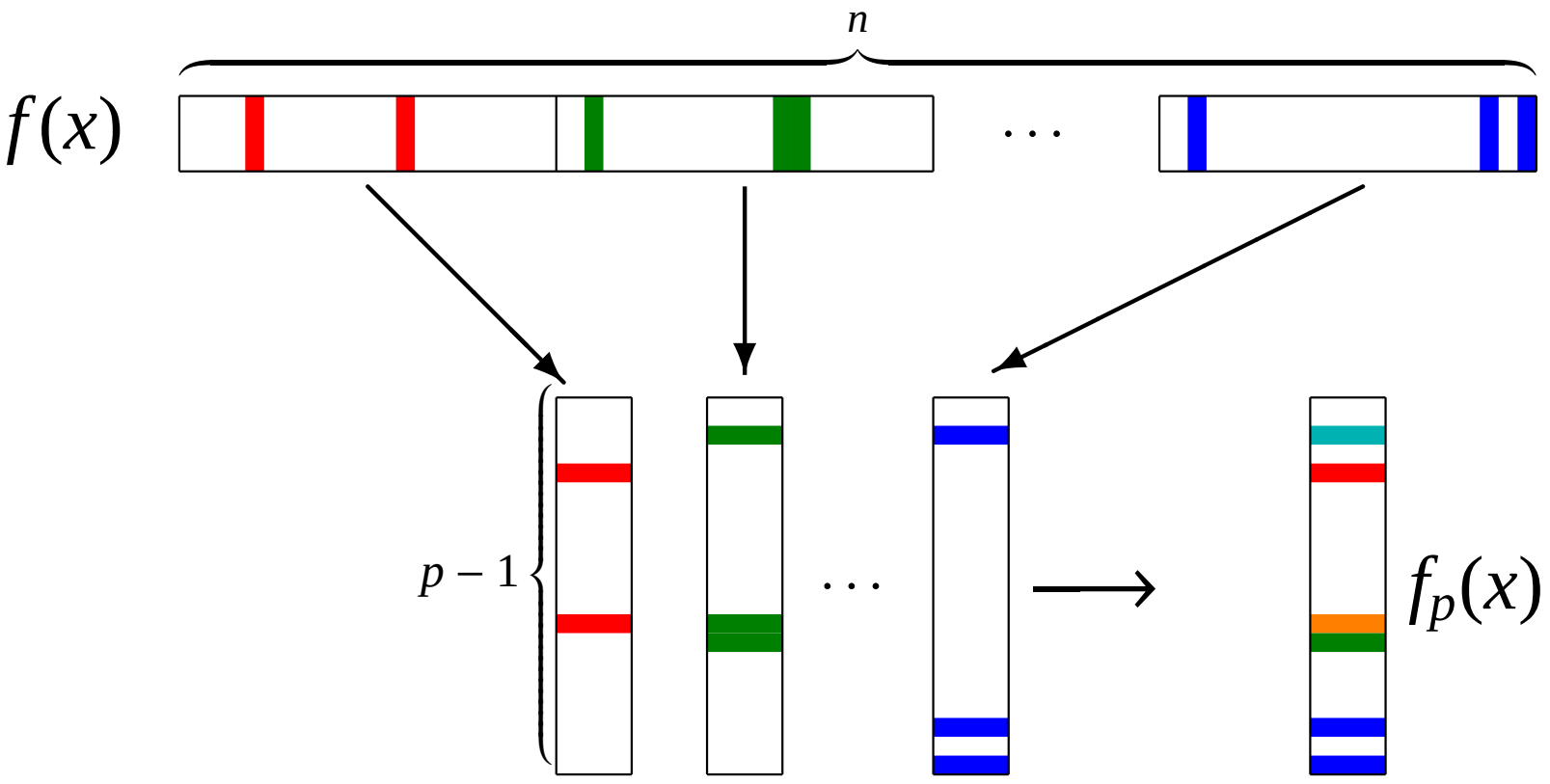
$$f(x) = c_1 (x-\alpha)^{e_1} + \cdots + c_t (x-\alpha)^{e_t}$$

$$f_p(x) = (c_1 \bmod p)(x-\alpha_p)^{e_1 \bmod (p-1)} + \cdots + (c_t \bmod p)(x-\alpha_p)^{e_t \bmod (p-1)},$$

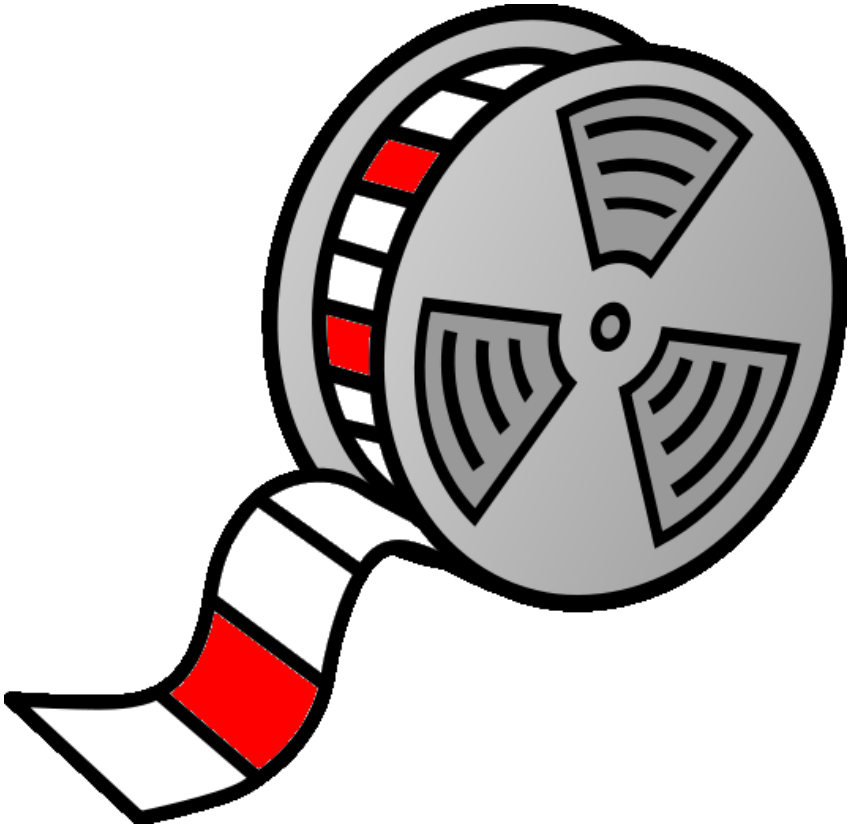
where $\alpha_p \equiv \alpha \bmod p$.

- $f(\theta) \bmod p = f_p(\theta \bmod p)$ whenever $\theta \not\equiv \alpha \bmod p$ (Fermat's Little Theorem)
- α_p is at least a t-sparse shift of $f_p(x)$

Pretty Picture #1



Pretty Picture #2



- Red squares indicate nonzero terms in the polynomial.
- The reel is the unit circle in $\mathbb{Z}_p$.

Outline of Algorithm

Input: Bound B on the bit length of the lacunary-shifted representation

1. Choose a prime p with $p \in O(B^{O(1)})$
2. Evaluate $f(1), \ldots, f(p-1) \bmod p$ to attempt to interpolate $f_p(x)$.
3. Use a dense sparsest shift method to compute α_p
4. Repeat Steps 1–3 enough times to recover α

Example

Unknown Polynomial in $\mathbb{Q}[x]$

$$f(x) = (x-3)^{107} - 485(x-3)^{54}$$

1. Choose a prime p with $p \in O(B^{O(1)})$

Step 1

$$p = 11$$

Example

Unknown Polynomial in $\mathbb{Q}[x]$

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1. Choose a prime p with $p \in O(B^{O(1)})$
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Step 2

$$f(1), \ldots, f(p-1) \bmod p = 10, 9, 0, 0, 2, 5, 2, 5, 10, 3$$

$$f_p(x) = x^7 + x^6 + 2x^5 + 9x^3 + 2x^2 + 8x + 9$$

Example

Unknown Polynomial in $\mathbb{Q}[x]$

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1. Choose a prime p with $p \in O(B^{O(1)})$
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Step 3

$$f_p(x) \equiv (x-3)^7 + 10(x-3)^4 \mod p$$

$$\alpha_p = 3$$

Example

Unknown Polynomial in $\mathbb{Q}[x]$

$$f(x) = (x-3)^{107} - 485(x-3)^{54}$$

1. Choose a prime p with $p \in O(B^{O(1)})$
2. Evaluate $f(1), \ldots, f(p-1) \bmod p$ to attempt to interpolate $f_p(x)$
3. Use a dense sparsest shift method to compute α_p
4. Repeat Steps 1–3 enough times to recover α

Step 4

$$\alpha_{11} = 3, \quad \alpha_{13} = 3, \quad \alpha_{17} = 3, \quad \ldots$$

$$\alpha = 3$$

Types of Failures

Two categories of failures:

- $f_p(x)$ is not computed correctly
- The sparsest shift of $f_p(x)$ is not α_p
  Equivalent to $\deg f_p(x) < 2t - 1$.

Next we develop sufficient conditions on p to avoid failure.

Exponents Vanish

$f_p(x)$ computed incorrectly

$$f(x) = 10(x-1)^{12} + 8(x-1)^3$$

$$p = 7$$

$$f_7(x) = 3(x-1)^0 + (x-1)^3$$

$$f(1) \bmod 7 = 0 \not\equiv 3 = f_7(1)$$

Condition: $(p-1) \nmid \max\{1, e_1\} \cdot e_2 \cdot e_3 \cdots e_t$

Test: Constant coeff. of computed $f_p(x)$ equals constant coeff. of $f(x)$ modulo p

Exponents Too Small

Sparsest shift of $f_p(x)$ is not α_p

$$f(x) = -4(x-2)^{145} + 14(x-2)^{26} + 3$$

$$p = 13$$

$$\begin{aligned} f_{13}(x) &= 9(x-2)^1 + (x-2)^2 + 3 \\ &\equiv (x-4)^2 + 12 \end{aligned}$$

Condition: $(p-1) \nmid e_t(e_t-1)(e_t-2)\cdots(e_t-(2t-2))$

Test: $\deg f_p(x) \geq 2B - 1 \geq 2t - 1$

Exponents Collide

Sparsest shift of $f_p(x)$ is not α_p

$$f(x) = 4(x-1)^{59} + 2(x-1)^{21} + 7(x-1)^{19} + 20$$

$$p = 11$$

$$\begin{aligned} f_{11}(x) &= 4(x-1)^9 + 2(x-1)^1 + 7(x-1)^9 + 9 \\ &= 2(x-1) + 9 \\ &\equiv 2(x-2) \end{aligned}$$

Condition: $(p-1) \nmid (e_t - e_1)(e_t - e_2)\cdots(e_t - e_{t-1})$

Test: $\deg f_p(x) \geq 2B - 1 \geq 2t - 1$

Coefficients Vanish

Sparsest shift of $f_p(x)$ is not α_p

$$f(x) = 69(x-5)^{42} - 12(x-5)^{23} + 4$$

$$p = 23$$

$$\begin{aligned} f_{23}(x) &= 0(x-5)^{20} + 11(x-5)^1 + 4 \\ &= 11(x-5) + 4 \\ &\equiv 11(x-13) \end{aligned}$$

Condition: $p \nmid c_t$

Test: $\deg f_p(x) \geq 2B - 1 \geq 2t - 1$

Sufficient Conditions

Definition

$$C = \max\{e_1, 1\} \cdot \prod_{i=2}^{t-1} e_i \cdot \prod_{i=1}^{t-1} (e_t - e_i) \cdot \prod_{i=0}^{2t-2} (e_t - i) \quad \leq 2^{4B^2}$$

Sufficient Conditions for Success

- $p \nmid c_t$
- $(p-1) \nmid C$

Approach

Choose primes $p = qk + 1$, for distinct primes q.
$q \mid (p-1)$, so $q \nmid C \Rightarrow (p-1) \nmid C$.

Choosing Primes Deterministically

1. Let $q_1, \ldots, q_k$ be the first k primes, for $k \in \tilde{O}(B^2)$.
2. Let p_i be the smallest prime s.t. $p_i \equiv 1 \bmod q_i$.
3. Set $\mathcal{P} = \{p_1, p_2, \ldots, p_k\}$.

Facts

- $|P| \in \Omega(B^2)$
- $p_i \in O(q_i^{5.5}) = \tilde{O}(B^{11})$ (Linnik, Heath-Brown '92)
- With ERH, $p_i \in \tilde{O}(q_i^2) = \tilde{O}(B^4)$ (Bach & Sorenson '96)

Probabilistic Algorithm

1. Choose a random prime $q \in O(B^2 \log B)$.
2. Choose random k's less than q until a prime $p = qk + 1$ is found.

We use Rousselet (1988) to show the density of primes of this type in the range $[q, q^2]$ is high, *once q is larger than some unknown constant*.

For this version, $p_i \in O(q_i^2) \in O\tilde{}(B^4)$.

Heuristic

Our analysis has been rather crude.
In fact, “most” primes will be good.

The following works well in practice:

1. Choose a random prime p in the range $[4B, 8B]$.

That’s it!

Complexity of Deterministic Algorithm

- Step 3 (computing sparsest shift of $f_p(x)$) dominates.

Bit Complexity

$$O(B^{78}(\log B)^{24} \log\log B)$$

Factor	Source
5.5	Bounds on Linnik's constant
2	Need $(p-1) \nmid C$ and $\log C \in O(B^2)$
7	Deterministic sparsest shift algorithm

This gives 77; one more factor from repeating $O(B)$ times.

Complexity of Probabilistic Algorithm

Bit Complexity

$$\tilde{O}(B^{17})$$

Factor	Source
2	Bounds on Linnik's constant using Rousselet
2	Need $(p-1) \nmid C$ and $\log C \in O(B^2)$
4	Probabilistic sparsest shift algorithm

Again, one more factor from $O(B)$ iterations.

Complexity of Heuristic

Bit Complexity

$$\tilde{O}(B^5)$$

This method gives us $p \in O(B)$ rather than $O(B^4)$ for probabilistic or $O(B^{11})$ for deterministic.

If we know $\alpha \in O(B)$, then we can just iterate once, for total cost $\tilde{O}(B^4)$.

Summary of Sparsest Shift Computation Techniques

Deterministic Algorithm

Actual complexity only $\tilde{O}(B^{29})$ unless ERH is false.

Probabilistic Algorithm

Always correct and (provably) probably much faster.

Heuristic

Faster in practice and provably never wrong, but might not terminate

Interpolation

Once α is known, we can construct a modular black box for evaluating $f(x + \alpha)$.

Then use lacunary interpolation along the lines of Kaltofen, Lakshman, & Wiley (1990) and Avendaño, Krick, & Pacetti (2006).

Conclusions

- Shifted-lacunary interpolation can be performed in polynomial time, for rational polynomials given by a modular black box.
- How to apply these techniques to other problems on lacunary polynomials?
- What about domains other than $\mathbb{Q}[x]$?
- What about multivariate rational polynomials?