Computing sparse multiples of polynomials

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Problem Statement

Sparsest Multiple Computation

Input Univariate polynomial f in $\mathbb{Q}[x]$ or $\mathbb{F}_q[x]$

Output A sparsest multiple of f of least degree.
(That is, a multiple of f with fewest nonzero terms)

Example (in $\mathbb{Q}[x]$)

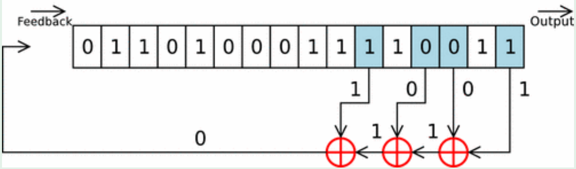
Input: $f = x^4 - 3x^3 + x^2 + 6x + 4$

Sparsest multiple: $h = x^{12} + 259x^6 + 64$

Not computed: $h/f = x^8 + 3x^7 + 8x^6 + 15x^5 + 15x^4 + \cdots \in \mathbb{Q}[x]$

Motivations: Cryptography

LFSR-based stream ciphers



- Sparse multiples of feedback polynomial lead to fast attacks.

TCHo

- Crytosystem proposed by (Aumasson et al, 2007)
- Computing sparse multiples in $\mathbb{F}_2[x]$ is their hard problem
- They explicitly assume average-case exponential lower bound

Motivations: Efficient Arithmetic

First noticed by Brent & Zimmerman (2003)

- Arithmetic in prime power fields $\mathbb{F}_{p^k}$ is usually done in $\mathbb{F}_p/\langle\Gamma\rangle$ for irreducible $\Gamma \in \mathbb{F}_p[x]$.
- Some field sizes admit no very-sparse irreducibles
- **Idea**: Work modulo a sparse multiple with low degree.
- Lots of searching has been done over $\mathbb{F}_2[x]$

Motivations: Computer Aided Geometric Design

source: CAEbridge, LLC

Motivations: Computer Aided Geometric Design

- Geometric surfaces are represented either
  - **Parametrically**: vector of parametric rational functions, or
  - **Implicitly**: solution set of multivariate polynomial
- Coverting from parametric to implicit is called implicitization.
- Can be thought of as finding a polynomial with given roots.
- Sparse implicitizatons make certain computations easier.

Motivations: Coding Theory

The problem can be formulated in linear algebra terms:

$$\begin{array}{ccccc} M_f & \times & v_g & = & v_h \\ \begin{bmatrix} f_0 & & & \\ f_1 & f_0 & & \\ \vdots & f_1 & \ddots & \\ f_d & \vdots & \ddots & f_0 \\ & f_d & \ddots & f_1 \\ & & \ddots & \vdots \\ & & & f_d \end{bmatrix} & & \begin{bmatrix} g_0 \\ g_1 \\ \\ \vdots \\ \\ g_{n-d} \end{bmatrix} & = & \begin{bmatrix} h_0 \\ h_1 \\ \\ \vdots \\ \\ h_n \end{bmatrix} \end{array}$$

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- Finding sparse vectors in a lattice is **NP**-hard.
- Similar to maximum likelihood decoding of cyclic codes.

Summary of Results

	Over $\mathbb{F}_q[x]$	Over $\mathbb{Q}[x]$
Binomial multiples	Equivalent to order finding	In **P**
t-sparse multiples, t constant	Harder than order finding	(mostly) in **P**
t-sparse multiples, t variable	???	???

Order Finding

Definition (Order)

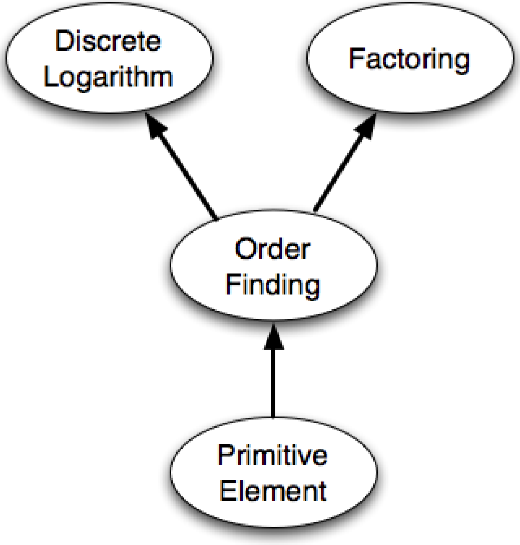
Given $\alpha \in \mathbb{F}_q[x]$, the multiplicative order of α is the least integer k such that $\alpha^k = 1$.

Connection to binomial multiples

If $f(\alpha) = 0$, and $f \mid (x^n - 1)$, then $\alpha^n = 1$.

- To show hardness, given $\alpha \in \mathbb{F}_{q^e}$, we take $f = (x-1) \cdot \mathrm{minpoly}(\alpha)$ in $\mathbb{F}_q[x]$ and find a binomial multiple of f.
- To show easiness, given $f \in \mathbb{F}_q[x]$ with $\deg f = d$, we find orders of all distinct roots $\alpha \in \mathbb{F}_{q^d}$ of f

Complexity of Order Finding



t-sparse Multiples Harder than Order Finding

Consider $\alpha \in \mathbb{F}_{q^e}$.
We use an oracle for t-sparse multiples to find the order of α.

- Find $g_i = \mathrm{minpoly}(\alpha^i) \in \mathbb{F}_q[x]$ for $i = 0, 1, \ldots, t-1$
- Let $f \in \mathbb{F}_q[x]$ be product of distinct g_i's.
- **Theorem**: f has a t-sparse multiple with degree $\leq n$
  iff $\mathrm{order}(\alpha) \leq n$.

  (In fact, the t-sparse multiple will be a binomial multiple.)

Sparse multiples in $\mathbb{Q}[x]$ connect to factorization

Related Problems

sparse multiple of a low-degree polynomial	$\Leftrightarrow$	low-degree factor of a sparse polynomial

- The latter problem has received much attention, both from mathematicians and computer scientists.

- It is convenient to associate with a squarefree input $f \in \mathbb{Q}[x]$ the roots $\theta_1, \theta_2, \ldots \in \bar{\mathbb{Q}}$ of its irreducible factors. Then f divides some $h \in \mathbb{Q}[x]$ iff $h(\theta_i) = 0$ for each i.

Binomial multiples in $\mathbb{Q}[x]$

Theorem (Risman (1976))

An irreducible $f \in \mathbb{Q}[x]$ with any binomial multiple has some binomial multiple of degree n, where $n = s \cdot t$ for some $s | \deg f$ and $\phi(t) | \deg f$.

- Leads to a polynomial *upper bound* on degree of least-degree binomial multiple
- Can then find binomial multiples of irreducibles by search
- For reducible f, correlating the binomial multiples of each factor just involves lcms and some more checks.
- We can generate examples of least-degree sparsest multiples with exponential degree and $\log$ height.

t-sparse Multiples in $\mathbb{Q}[x]$

Key Tool: Lenstra (1999)

If $h \in \mathbb{Q}[x]$ written $h_1 + x^k h_2$ has a big gap: $k \gg h_2$,
then any low-degree non-cyclotomic factor of h
is a factor of both h_1 and h_2

For instance, consider the polynomial h given by:

$$\underbrace{2x^{105} + 3x^{104} - 2x^{103} - x^{102} + x^{101} - 3x^{100}}_{h_2} \underbrace{\qquad\qquad\qquad\qquad}_{\text{(the gap)}} \underbrace{-2x^4 + x^3 + 4x^2 - 3x}_{h_1}$$

$f = 2x^2 + x - 3$ divides h, so $f | h_1$ and $f | h_2$.

- Lenstra used this for lacunary factorization

Gap theorem for sparsest multiples

Turning the gap theorem around, we get:

Theorem

The least-degree t-sparse multiple with height at most c of a non-cyclotomic polynomial $f \in \mathbb{Q}[x]$ has degree bounded by

$$(t + \log c + \deg f)^{O(1)}$$

With such a degree bound, the problem reduces to finding the least-height t-sparse rational vector in a lattice.

This is polynomial-time when t is constant using (Ajtai, Kumar, and Sivakumar 2001).

Handling cyclotomics: Example

$$f = x^{10} - 5x^9 + 10x^8 - 8x^7 + 7x^6 - 4x^5 + 4x^4 + x^3 + x^2 - 2x + 4$$

Step 1: Extract cyclotomic factors

$$f = \underbrace{(x^2 - x + 1)}_{\Phi_6} \cdot \underbrace{(x^4 - x^3 + x^2 - x + 1)}_{\Phi_{10}} \cdot \underbrace{(x^4 - 3x^3 + x^2 + 6x + 4)}_{\substack{f_D \\ \text{(cyclotomic-free)}}}$$

Handling cyclotomics: Example

$$f = x^{10} - 5x^9 + 10x^8 - 8x^7 + 7x^6 - 4x^5 + 4x^4 + x^3 + x^2 - 2x + 4$$

Step 2: Calculate degree bound

Target sparsity: ≤ 10, target height: ≤ 1000

Actual degree bound: $\deg h \leq 11\,195\,728$
(asymptotically polynomial, practically quite large!)

For this example, we'll cheat and say $\deg h \leq 20$

Handling cyclotomics: Example

$$f = x^{10} - 5x^9 + 10x^8 - 8x^7 + 7x^6 - 4x^5 + 4x^4 + x^3 + x^2 - 2x + 4$$

Step 3: Find low-degree sparsest multiples

Sparsest multiple of f with degree ≤ 20:

$$h_A = x^{11} - 3x^{10} + 12x^8 - 9x^7 + 10x^6 - 4x^5 + 9x^4 + 3x^3 + 8$$

Sparsest multiple of f_D (cyclotomic-free part):

$$h_B = x^{12} + 259x^6 + 64$$

Handling cyclotomics: Example

$$f = x^{10} - 5x^9 + 10x^8 - 8x^7 + 7x^6 - 4x^5 + 4x^4 + x^3 + x^2 - 2x + 4$$

Step 4: Sparsest multiple of cyclotomic part

Recall $f = \Phi_6 \cdot \Phi_{10} \cdot f_D$.
Cyclotomic part is $f_C = \Phi_6 \cdot \Phi_{10}$

Sparsest multiple of f_C:

$$h_C = \left(x^{\mathrm{lcm}(6,10)} - 1\right) = \left(x^{30} - 1\right)$$

Handling cyclotomics: Example

$$f = x^{10} - 5x^9 + 10x^8 - 8x^7 + 7x^6 - 4x^5 + 4x^4 + x^3 + x^2 - 2x + 4$$

Step 5: Compare candidates

Two candidates for sparsest multiple of f:

- $h_A = x^{11} - 3x^{10} + 12x^8 - 9x^7 + 10x^6 - 4x^5 + 9x^4 + 3x^3 + 8$
- $h_B \cdot h_C = x^{42} + 259x^{36} + 64x^{30} - x^{12} - 259x^6 - 64$

Conclusion: A sparsest multiple of f is

$$h = x^{42} + 259x^{36} + 64x^{30} - x^{12} - 259x^6 - 64$$

Open Problems

- Proving **NP**-hardness for the general case (t variable) over rationals or finite fields
- Improving the t-sparse algorithm over $\mathbb{Q}[x]$:
  - More practical degree bounds
  - Eliminate need for a priori height bound
  - De-randomize
  - Handle missing case:
    non-cyclotomic and repeated cyclotomic factors
- Extending to multivariate polynomials