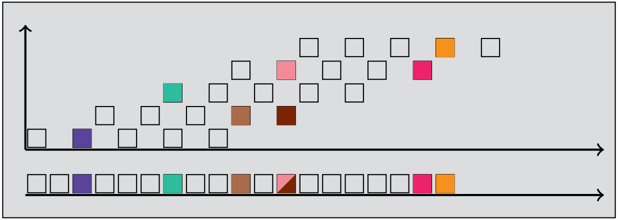
Multivariate sparse interpolation using randomized Kronecker substitutions



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Overview

Our Main Result

A new **randomization** that improves
the **Kronecker substitution** trick
by **reducing the degree**
when the polynomial is **sparse**.

The initial application is sparse interpolation.

Kronecker

Definition

The *Kronecker Substitution* (1882) is a map:

multivariate polynomial → univariate polynomial

$$\mathsf{R}[x, y] \to \mathsf{R}[z]$$

defined by $f(x, y) \mapsto f(z, z^D)$, where $D > \deg_x(f)$.

This map is a homomorphism and
it is invertible given the degree bound D.

(Can also map polynomials to integers, or multivariate to univariate.)

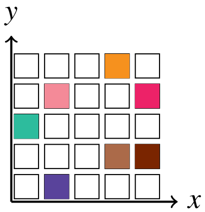
Kronecker Example

Example

$f(x, y) = \blacksquare x + \blacksquare x^3 y + \blacksquare x^4 y + \blacksquare y^2 + \blacksquare x y^3 + \blacksquare x^4 y^3 + \blacksquare x^3 y^4$

(colored boxes ■ represent coefficients in R)

Representation of $f(x, y)$:



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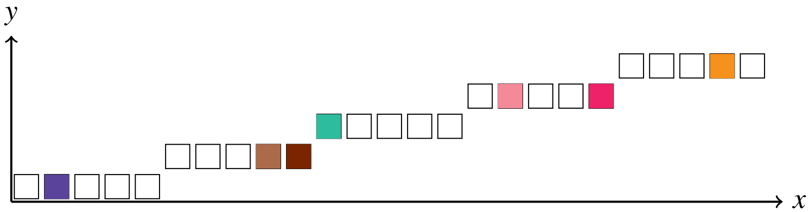

Kronecker Example

Example

$$f(x,y) = \blacksquare x + \blacksquare x^3 y + \blacksquare x^4 y + \blacksquare y^2 + \blacksquare xy^3 + \blacksquare x^4 y^3 + \blacksquare x^3 y^4$$

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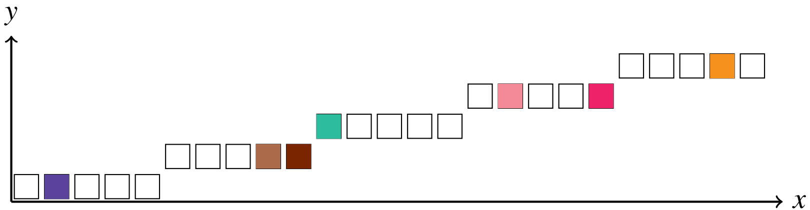
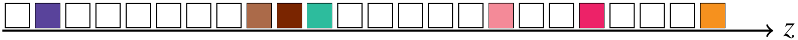


Kronecker substitution: $f(z, z^D)$, degree 23



Applications of Kronecker

1. Multiplication

The Kronecker substitution is often used to multiply polynomials.

- Reducing $\mathbb{Z}[x]$ to $\mathbb{Z}$ (Schönhage '82, Harvey '09)
- Exponent packing for multivariate sparse polynomials (Monagan & Pearce '07)
- Reducing multivariate dense to bivariate multiplilcation (Moreno Maza & Xie '11)

Applications of Kronecker

2. Factorization

Kronecker substitutions can be used to discover the factorization of multivariate polynomials.

- Kronecker's original motivation! (1882)
- Reducing multivariate to bivariate factorization (Kaltofen 1982)
- Computing perfect roots of sparse polynomials (Giesbrecht & R. '11)

Randomized Kronecker substitution

Let $f \in R[x, y]$ with $\deg_x(f) = d_x$, $\deg_y(f) = d_y$ and $d_x, d_y < D$.

The Idea

Instead of a usual Kronecker substitution:

$$f(x, y) \mapsto f(z, z^D)$$

we choose random integers p, q and the homomorphism:

$$f(x, y) \mapsto f(z^p, z^q)$$

(Note: similar trick to Klivans & Spielman '01)

Challenge: How to choose p, q
so that f can be recovered from $f(z^p, z^q)$?

Randomized Kronecker substitution

Let $f \in \mathrm{R}[x, y]$ with $\deg_x(f) = d_x$, $\deg_y(f) = d_y$ and $d_x, d_y < D$.

The Idea

Instead of a usual Kronecker substitution:

$$f(x, y) \mapsto f(z, z^D) \qquad \longrightarrow \text{degree } d_x + Dd_y \approx D^2$$

we choose random integers $p, q \ll D$ and the homomorphism:

$$f(x, y) \mapsto f(z^p, z^q) \qquad \longrightarrow \text{degree } d_x p + d_y q \ll D^2$$

(Note: similar trick to Klivans & Spielman '01)

Challenge: How to choose p, q as small as possible
so that f can be recovered from $f(z^p, z^q)$?

Randomized Kronecker Example

Example

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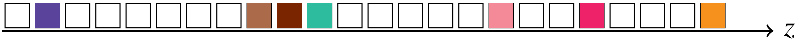

Randomized Kronecker Example

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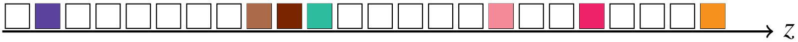
Representation of $f(x^2, y)$**:**

Kronecker substitution: $f(z, z^D)$, degree 23



Randomized Kronecker Example

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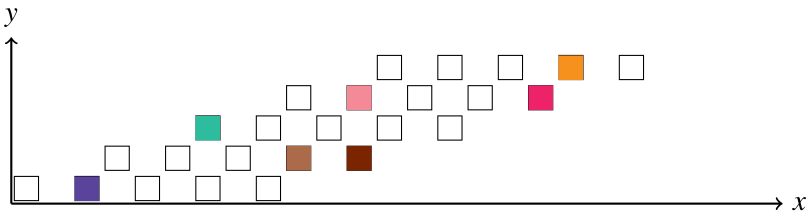
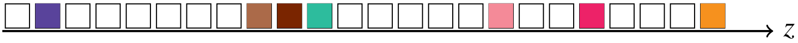
Representation of $f(x^2, x^3y)$**:**



Kronecker substitution: $f(z, z^D)$, degree 23



Randomized Kronecker Example

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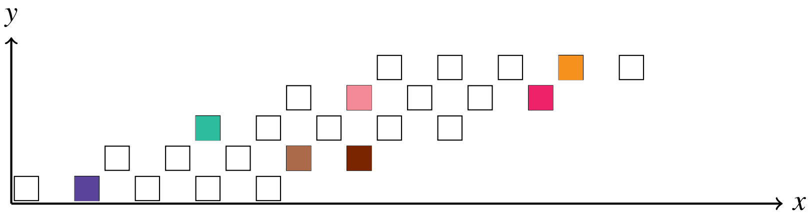
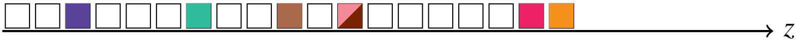
Representation of $f(x^2, x^3y)$:



Randomized Kronecker substitution: $f(z^2, z^3)$, degree 18



Less trivial example

Example

$$f = (x^{50} - x^{35} + x^{23} - 1) \circ (x^{127}y^2 + xy^{127} + x^3y^{102} + x^7y^{77} + x^{45}y^{27} + x^{17}y^{52})$$

This polynomial has $\deg_x = \deg_y = 6350$ and $\#f = 161\,778$ nonzero terms over $\mathbb{F}_{13}[x, y]$.

The usual Kronecker substitution $f(z, z^{6351})$ has degree $40\,328\,900$.

■=161 778 nonzero coeffs, □=40 167 122 zero coeffs

The substitution $f(z^{101}, z^{103})$ has degree $659\,100$ (61x smaller):

■=148 558 nonzero coeffs, ■=**6610 collisions**, □=503 932 zero coeffs

Probabilistic analysis, bivariate case

- We choose exponents p, q to be primes in this case, and evaluate the map $f(x,y) \mapsto f(z^p, z^q)$
- How large should p, q be to minimize collisions?

Theorem

Suppose $f \in \mathsf{R}[x,y]$ has degree $< D$ and at most T nonzero terms. If p, q are randomly chosen primes of size $O(\sqrt{T}\log D)$, then w.h.p. there will be fewer than $T/2$ collisions.

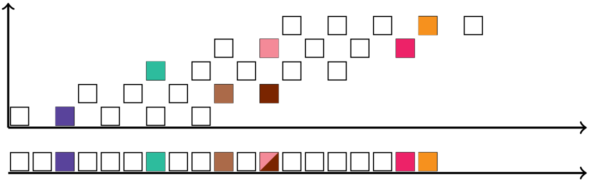
Proof trick:

If $z^{a_i p} z^{b_i q} = z^{a_j p} z^{b_j q}$, then
$(a_i - a_j)p = (b_j - b_i)q$, so
$p | (b_i - b_j)$ and $q | (a_i - a_j)$.

The two independent divisibility conditions give the $\sqrt{T}$ term in the size of the primes.

Challenges of randomized Kronecker

With the benefit of a smaller degree, comes two challenges:



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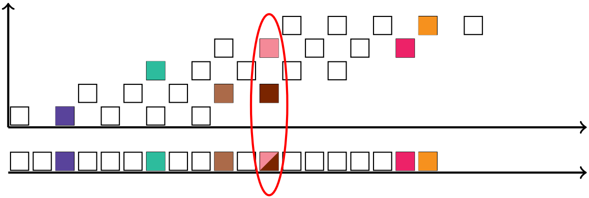
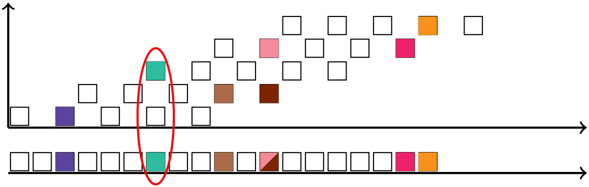


Challenges of randomized Kronecker

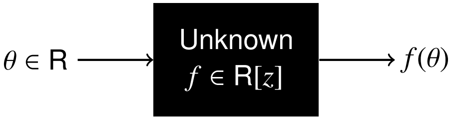
With the benefit of a smaller degree, comes two challenges:

- There will be some collisions of terms
- The map is no longer invertible



The way around these will depend on the application.

Background: Univariate Interpolation



Problem: determine the coefficients and exponents of f

Two flavors of univariate interpolation

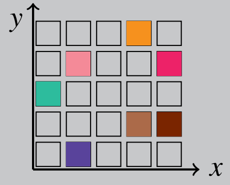
Say $\deg f < D$ and $\#f < T$.

- **Dense**: Requires D probes and $O(D \log D)$ computation. (Newton, Waring, Lagrange, FFT)
- **Supersparse**: Requires $O(T)$ probes and $O(T \log^2 D)$ computation. (Prony, Ben-Or & Tiwari '89, Garg & Schost '09)

More background: Zippel Interpolation

(Zippel '79, Kaltofen/Lee/Lobo '00)

Idea: Do a random projection to univariate, then interpolate up from each nonzero coefficient.



Total cost:

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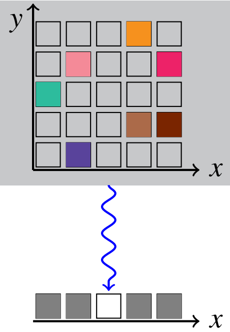

Total cost: 1 univariate interpolation, degree D

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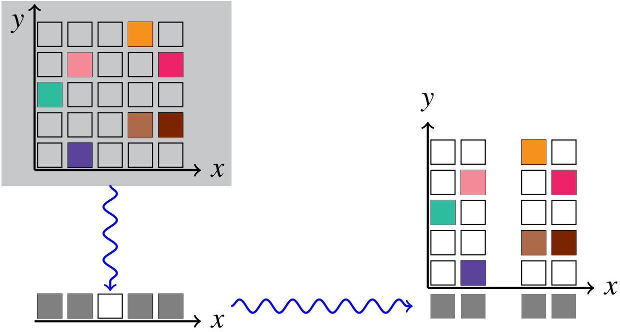
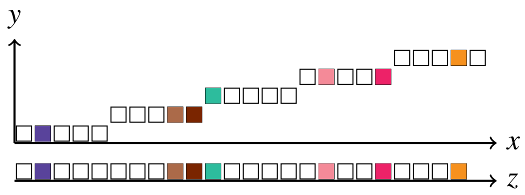


Total cost: At most $t + 1$ univariate interpolations, each degree D

Applications of Kronecker

3. Interpolation

Kronecker can also reduce multivariate to univariate interpolation.



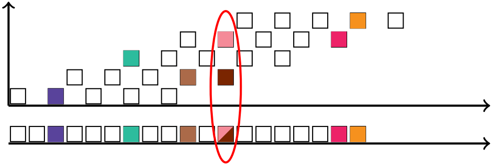
1. Evaluate $f(\theta, \theta^D)$ for many univariate evaluation points θ
2. Use univariate interpolation to discover $f(z, z^D)$
3. Invert the map to discover $f \in R[x, y]$

(Kaltofen, Lakshman, Wiley '90; Kaltofen & Lee '03; Javadi & Monagan '10; van der Hoeven & Lecerf '13)

Our method for interpolation

We use the same idea, but must address the two challenges:

- There will be some collisions of terms



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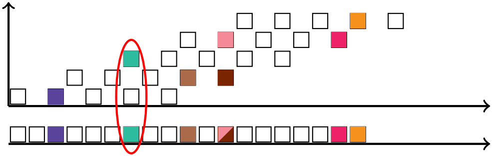

Our method for interpolation

We use the same idea, but must address the two challenges:

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Our theorem guarantees *most* terms do not collide.
Use the technique from (A., Giesbrecht, R. '13) and
iterate $O(\log T)$ times

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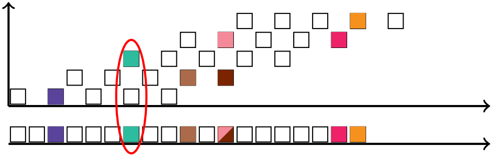

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Get every term in at least two reductions, then solve:

$$\begin{bmatrix} p_1 & q_1 \\ p_2 & q_2 \end{bmatrix} \begin{bmatrix} e_x \\ e_y \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix},$$

where u, v are the exponents in the two univariate images.

Multi- to Uni-variate methods comparison

n=# of variables, D=degree bound, T=sparsity bound

	# of reductions	degree of each
Kronecker '82	1	D^n
Zippel '88	nT	D
Klivans & Spielman '01	n	$O(n^2T^2D)$
Ours (bivariate)	$O(\log T)$	$O(\sqrt{T} \log D)$
Ours (≥ 3 variate)	$O(n + \log T)$	$O(TD)$

Multivariate interpolation complexity

n=# of variables, D=degree bound, T=sparsity bound

Using **dense** univariate interpolation

	# of probes *and* computation cost
Kronecker	D^n
Zippel	nTD
Ours (bivariate)	$\sqrt{T}D$
Ours (≥ 3 variate)	nTD

(All costs are soft-oh, ignoring logarithmic factors.)

Multivariate interpolation complexity

n=# of variables, D=degree bound, T=sparsity bound

Using **supersparse** univariate interpolation

	# of probes	computation cost
Kronecker	T	$n^2 T \log^2 D$
Zippel	nT^2	$nT^2 \log^2 D$
Ours (bivariate)	T	$T \log^2 D$
Ours (≥ 3 variate)	nT	$nT \log^2 D$

(All costs are soft-oh, ignoring logarithmic factors.)

Did I mention multivariate?

The trick with primes does not work when $n \geq 3$.

In this case we choose random integer exponents, but have a somewhat weaker result:

Theorem

Suppose $f \in \mathsf{R}[x_1, x_2, \ldots, x_n]$ *has degree* $< D$ *and at most* T *nonzero terms.*
If $s_1, s_2, \ldots, s_n$ *are random integers of size* $O(T)$*, then w.h.p. there will be fewer than* $T/2$ *collisions in the substitution*

$$f(z^{s_1}, z^{s_2}, \ldots, z^{s_n}).$$

Proof idea: Any vector $(s_1, \ldots, s_n)$ that makes two terms collide must lie in some $(n-1)$-dimensional null space.

Future work

- Strengthen probabilistic analysis, especially in the multivariate case
- Work on implementation
- Apply theoretical results to more problems
- Can we do better when we know (some of) the structure?

Thanks!

Back to example
Back to results