Faster Sparse Interpolation over Finite Fields and Complex Numbers

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The Problem

The basic **sparse interpolation** problem is as follows: Given a **black box** (i.e. way to evaluate) an unknown polynomial

$$f = c_1 x^{e_1} + c_2 x^{e_2} + \cdots + c_t x^{e_t},$$

determine the coefficients c_i and exponents e_i.

We are interested in two cases:

- Coefficients come from a large, unchosen finite field
- Coefficients are approximations to complex numbers

Remainder Black Box

A **remainder black box** takes a monic polynomial g and evaluates f rem g.

Example: unknown polynomial is

$$f = 5x^6 - 20x^{139} + 16x^{218} - 3x^{381}.$$

Given $g = x^{10} - 1$, the black box returns

$$f \text{ rem } g = -3x + 5x^6 + 16x^8 - 20x^9.$$

Observe: exponents reduced modulo 10.

Garg and Schost's Algorithm

Garg & Schost (TCS 2009): first polynomial-time algorithm for sparse interpolation over a large, unchosen finite field.

Overview: Given remainder black box for unknown

$$f = c_1 x^{e_1} + c_2 x^{e_2} + \cdots + c_t x^{e_t},$$

define the unknown integer polynomial

$$\Gamma(z) = (z - e_1)(z - e_2)\cdots(z - e_t) \in \mathbb{Z}[z].$$

For primes $p \in O(t^2 \log \deg f)$, evaluate f rem $x^p - 1$.
This gives us the set $\{e_1 \text{ rem } p, e_2 \text{ rem } p, \ldots e_t \text{ rem } p\}$, from which the coefficients of Γ mod p can be computed.

Repeating $O(t^2 \log d)$ times gives the coefficients of Γ, and we perform root finding over $\mathbb{Z}[z]$ to find the exponents e_i.

Diversification

- We call a polynomial with all coefficients distinct **diverse**.
- Diverse polynomials are easier to interpolate.
- We use randomization to create diversity.

Theorem. *If $q \gg t^2 \deg f$, $f \in \mathbb{F}_q[x]$, and $\alpha \in \mathbb{F}_q$ is chosen randomly, then $f(\alpha x)$ is probably diverse.*

Theorem. *If $f \in \mathbb{C}[x]$ has large coefficients and ζ is an order-$O(t^2)$ root of unity, $f(\zeta x)$ is probably diverse.*

Diversity in the latter case (approximate) means sufficiently separated coefficients.

Example over finite field $\mathbb{F}_{101}$

Let $f = 57 + 5x^{74} + 57x^{76} + 5x^{92} \in \mathbb{F}_{101}[x]$ be unknown. Note that f is *not* diverse.

Diversify. Randomly choose $\alpha \in \mathbb{F}_{101}$: $\alpha = 21$.
Also choose $p_1 \in O(t^2 \log \deg f)$: $p_1 = 11$, and evaluate

$$f(\alpha x) \text{ rem} (x^{11} - 1) = 57 + x^4 + 19x^8 + 15x^{10}.$$

This gives sparsity $t = 4$ and shows that $f(\alpha x)$ is diverse.

Further evaluations. Let $p_2 = 5$ and $p_3 = 7$. Evaluate

$$f(\alpha x) \text{ rem} (x^5 - 1) = 57 + 15x + x^2 + 19x^4$$
$$f(\alpha x) \text{ rem} (x^7 - 1) = 57 + x + 19x^4 + 15x^6.$$

Recover exponents. Because we know $p_1 p_2 p_3 > \deg f$, like terms are correlated **using the diverse coefficients**, and then exponents are found by Chinese remaindering:

$$e_1 = 0, \quad e_2 = 74, \quad e_3 = 76, \quad e_4 = 92.$$

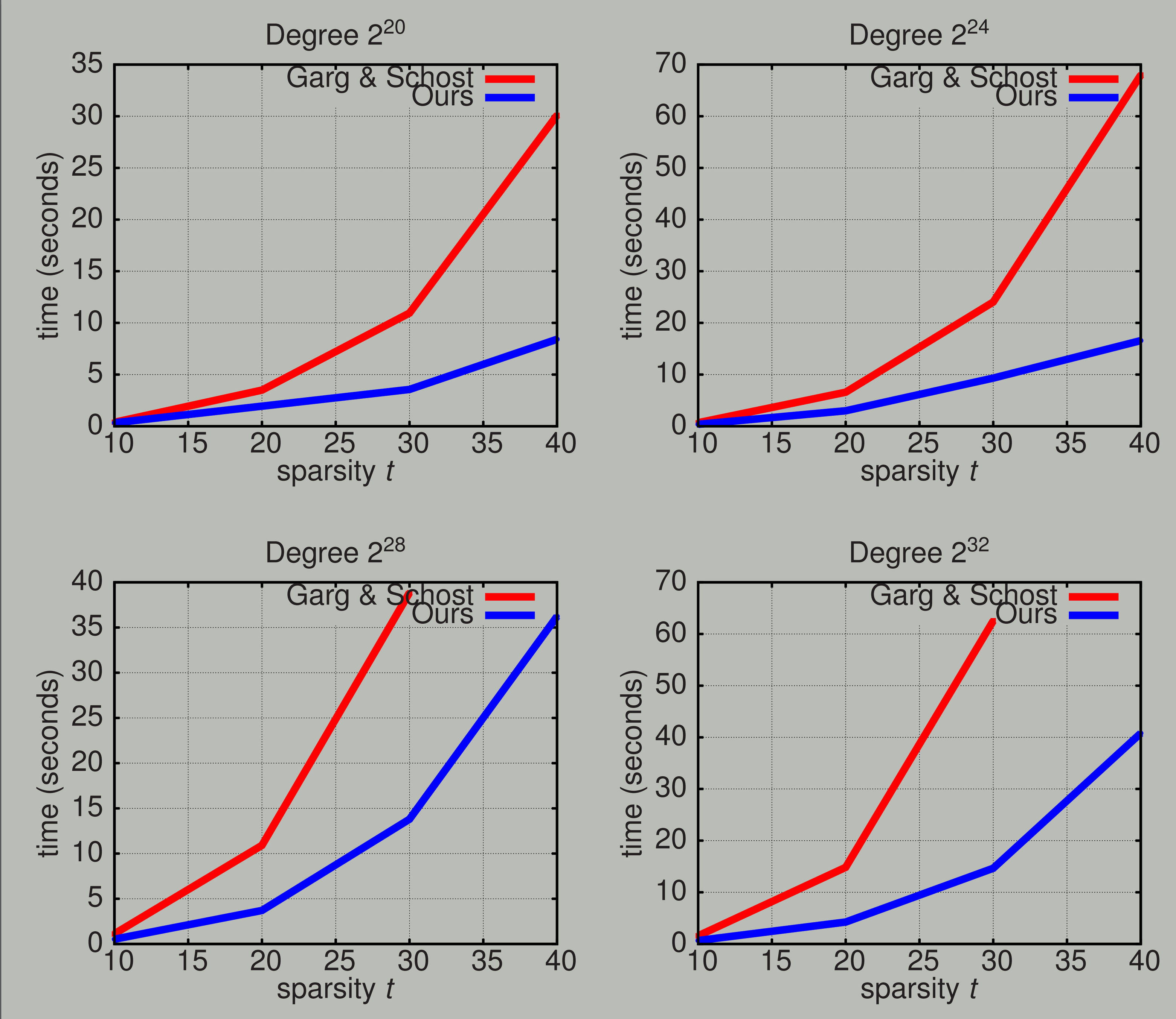
Recover coefficients. Once we know the exponents, the coefficients are determined from any modular evaluation.

Summary of results

Finite fields: Randomized cost is $\tilde{O}(t^2 \log^2 \deg f)$.

Approximate: In the same time, and with ϵ noise, we can compute a $g \in \mathbb{C}[x]$ such that $\|f - g\|_2 < \epsilon \|f\|_2$.

Finite field implementation experiments



Experimental stability in approximate algorithm

Noise	Mean Error	Median Error	Max Error
0	4.440 e−16	4.402 e−16	8.003 e−16
$\pm 10^{-12}$	1.113 e−14	1.119 e−14	1.179 e−14
$\pm 10^{-9}$	1.149 e−11	1.191 e−11	1.248 e−11
$\pm 10^{-6}$	1.145 e−8	1.149 e−8	1.281 e−8

Extending to multivariate

Now consider an unknown multivariate $f \in \mathsf{F}[x_1, \ldots, x_n]$.
We can perform sparse interpolation in one of two ways:

Kronecker substitution. Consider the polynomial

$$\hat{f} = f(y, y^d, y^{d^2}, \ldots, y^{d^{n-1}}).$$

If $d > \deg_{x_i} f$ for all i, then the terms of the *univariate* polynomial $\hat{f}$ correspond to those of f.

Zippel's method. Zippel's multivariate interpolation algorithm can be hybridized with our univariate algorithms. The method is randomized and works variable-by-variable, resulting in more univariate calls with lower degrees.