Complexity of Shifted-Lacunary Polynomial Interpolation

Daniel S. Roche

ORCCA
Ontario Research Centre for Computer Algebra

Symbolic Computation Group
School of Computer Science
University of Waterloo

University of Waterloo

SIG Theory and Complexity Seminar
University of Delaware
9 January 2009

This is joint work
with Mark Giesbrecht.

First presented at the
MACIS conference
in Paris, France,
December 5–7, 2007.

Complexity of Shifted-Lacunary Polynomial Interpolation

General Problem

Determining a function from its values.

Goals

- Find the simplest possible formula.
- Don’t take too long.

Necessities

- What type of function? (output type)
- How big can it be? (output size)

Complexity of Shifted-Lacunary Polynomial Interpolation

Example

$$f = (x-3)^{107} - 485(x-3)^{54}$$

Suppose we can evaluate $f(\theta)$ at any chosen point θ.

- Can we find a formula for f?

Complexity of Shifted-Lacunary Polynomial Interpolation

Example

$$f = (x-3)^{107} - 485(x-3)^{54}$$

Suppose we can evaluate $f(\theta)$ at any chosen point θ.

- Can we find a simple formula for f?

Complexity of Shifted-Lacunary Polynomial Interpolation

Example

$$f = (x-3)^{107} - 485(x-3)^{54}$$

Suppose we can evaluate $f(\theta)$ at any chosen point θ.

- Can we find a simple formula for f in a reasonable amount of time?

Complexity of Shifted-Lacunary Polynomial Interpolation

Dense Methods

Definition (Dense Representation)

$$f = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n,$$

where $n = \deg(f)$ and $a_0, a_1, \ldots, a_n \in \mathsf{R}$

- Studied by Newton (1711), Waring (1779), . . .
- Highly efficient implementations available

Complexity of Shifted-Lacunary Polynomial Interpolation

Dense Methods

Definition (Dense Representation)

$$f = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n,$$

where $n = \deg(f)$ and $a_0, a_1, \ldots, a_n \in \mathrm{R}$

Example

For $f = (x-3)^{107} - 485(x-3)^{54}$, we will have

$$f = x^{107} - 321x^{106} + 51039x^{105} - 5359095x^{104} + \cdots$$
$$+ 4020099274965907985483758515231179267430359014481937 3x$$
$$- 1127130637840908780976768693419860197828458989848152$$

This is way too big! (twice exponential in the desired size)

Complexity of Shifted-Lacunary Polynomial Interpolation

(Lacunary polynomials are sometimes called sparse or supersparse.)

- Default representation in Maple, Mathematica, etc.
- Some things are hard (Plaisted 1977, 1984)
- Some things aren't: Interpolation, finding low-degree factors
- Some things are unknown!

Complexity of Shifted-Lacunary Polynomial Interpolation

Sparse Methods

Definition (Lacunary Representation)

$$f = b_0 + b_1 x^{d_1} + b_2 x^{d_2} + \cdots + b_s x^{d_s},$$

where $d_1 < d_2 < \cdots < d_s = n$ and $b_1, \ldots, b_s \in \mathsf{R} \setminus \{0\}$

- Baron de Prony (1795), Ben-Or & Tiwari (1988), Kaltofen, Lakshman, Wiley, Lee, Lobo, ...
- Need to choose evaluation points
- R must have a high-order element and a fast logarithm.

Complexity of Shifted-Lacunary Polynomial Interpolation

Sparse Methods

Definition (Lacunary Representation)

$$f = b_0 + b_1 x^{d_1} + b_2 x^{d_2} + \cdots + b_s x^{d_s},$$

where $d_1 < d_2 < \cdots < d_s = n$ and $b_1, \ldots, b_s \in \mathsf{R} \setminus \{0\}$

Example

If $f = (x-3)^{107} - 485(x-3)^{54}$,
this helps iff we know the sparsest shift 3,
since $f(x+3) = x^{107} - 485x^{54}$ is 2-sparse.

Complexity of Shifted-Lacunary Polynomial Interpolation

Definition (Shifted-Lacunary Representation)

$$f = c_0 + c_1(x-\alpha)^{e_1} + c_2(x-\alpha)^{e_2} + \cdots + c_t(x-\alpha)^{e_t},$$

where $e_1 < \cdots < e_t = n$ and t is minimal for any α

- This is our problem.
- Can be reduced to finding the sparsest shift α.
- We restrict the domain to $\mathbb{Q}[x]$.
- No previous polynomial-time algorithm known.

Complexity of Shifted-Lacunary Polynomial Interpolation

We give an algorithm with
output-sensitive polynomial-time complexity,
specifically, *bit complexity* polynomial in:

- Number of nonzero terms t
- Logarithm of the degree n
- Size of the coefficients $c_1, \ldots, c_t$
- Size of the sparsest shift α

Black box calls are assumed to have constant cost.

Computing the Sparsest Shift

- Borodin & Tiwari (1991)
  Compute sparsest shift from evaluation points (open)
- Grigoriev & Karpinski (1993)
  Compute sparsest shift from a black-box function.
  State need for complexity *not* polynomial in n
- Lakshman & Saunders (1996)
  Compute sparsest shift from dense representation
- Giesbrecht, Kaltofen, Lee (2003)
  Current best results (deterministic & probabilistic)

Uniqueness and Rationality of Sparsest Shift

Theorem (Lakshman & Saunders (1996))

If the degree is at least twice the sparsity,
then the sparsest shift is unique and rational.

Example

$$f = (x-3)^{107} - 485(x-3)^{54}$$

$\Rightarrow$ 3 is the only shift with ≤ 54 terms

Condition not satisfied means polynomial is dense.

Black Box Model

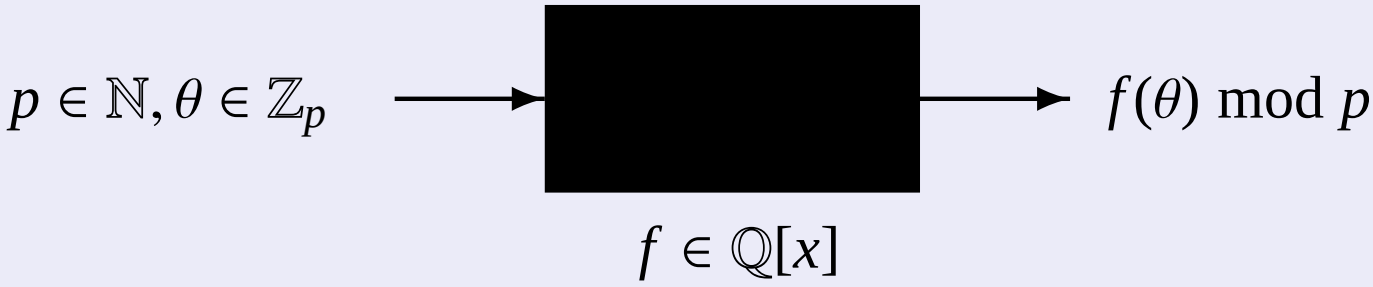
Arbitrary evaluations will usually be very large:

Example

$$f = (x-3)^{107} - 485(x-3)^{54}$$
$$f(1) = -162259276829222100374855109050368$$

To control evaluation size, use modular arithmetic:

The “Modular Black-Box”



Modular Reductions

Definition (Rational remainder)

$a \operatorname{rem} m = b$ iff $a \equiv b \bmod m$ and $0 \leq b < m$.

Definition (Shifted remainder)

$a \operatorname{rem}_1 m = b$ iff $a \equiv b \bmod m$ and $1 \leq b \leq m$.

Definition (Modular-reduced Polynomial)

For $f \in \mathbb{Q}[x]$, $f^{(p)}$ is the unique polynomial in $\mathbb{Z}_p[x]$ with degree less than p such that $f \equiv f^{(p)} \bmod (x^p - x)$.

Modular-Reduced Polynomial

Definition

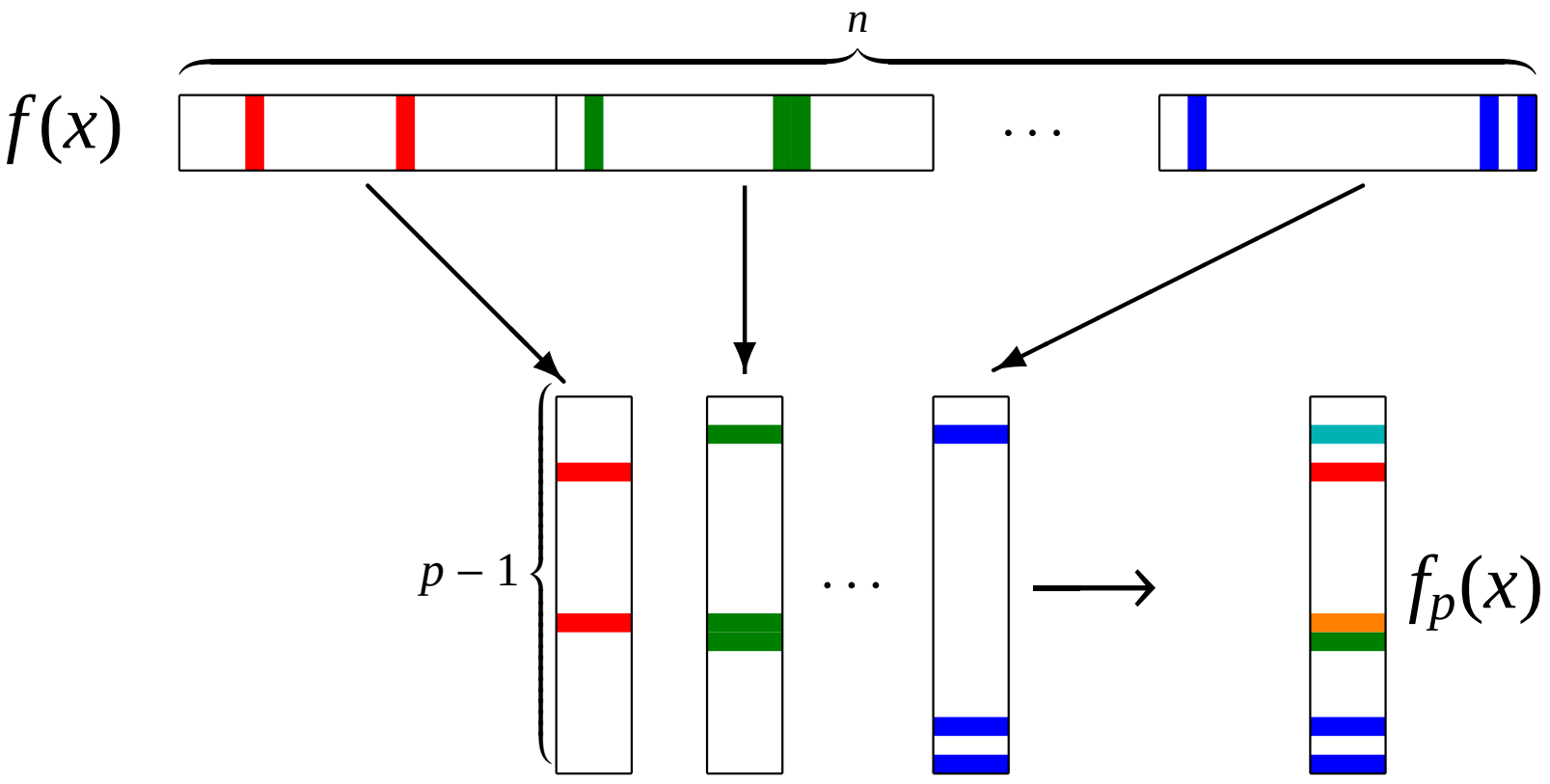
$$f = c_0 + c_1 (x-\alpha)^{e_1} + \cdots + c_t (x-\alpha)^{e_t}$$

$$f^{(p)} = (c_0 \operatorname{rem} p) + (c_1 \operatorname{rem} p)(x-\alpha_p)^{e_1 \operatorname{rem}_1 (p-1)} + \cdots + (c_t \operatorname{rem} p)(x-\alpha_p)^{e_t \operatorname{rem}_1 (p-1)},$$

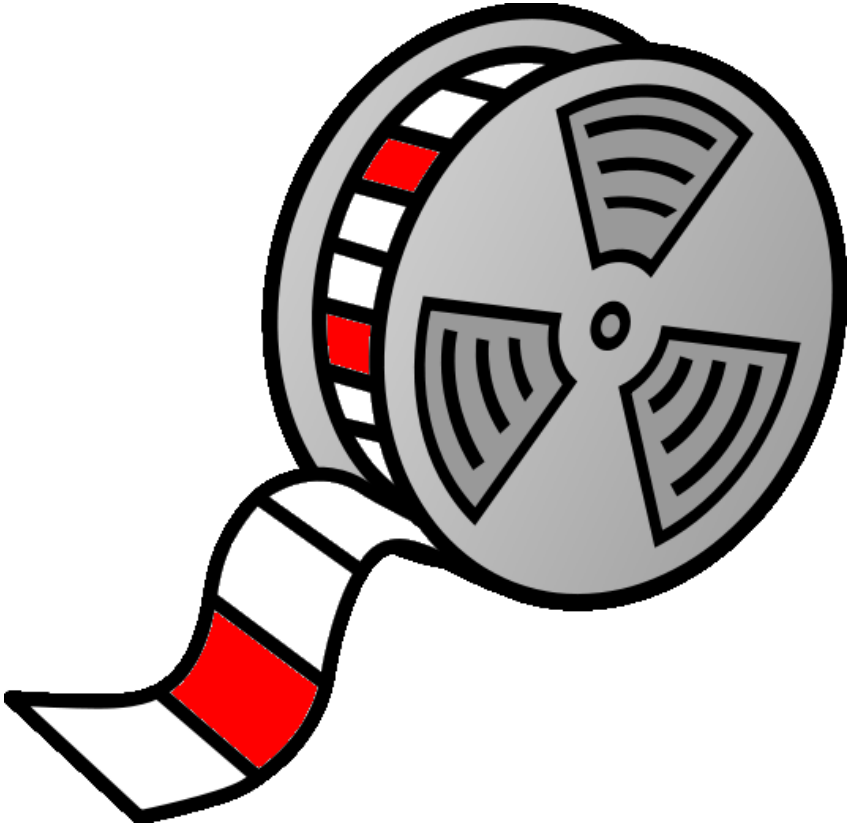
where $\alpha_p \equiv \alpha \bmod p$.

- $f(\theta) \operatorname{rem} p = f^{(p)}(\theta \operatorname{rem} p), \qquad \forall \theta \in \mathbb{Z}$
  (Fermat's Little Theorem)
- α_p is at least a t-sparse shift of $f^{(p)}$

Pretty Picture #1



Pretty Picture #2



- Red squares indicate nonzero terms in the polynomial.
- The reel is the unit circle in $\mathbb{Z}_p$.

Outline of Algorithm

Input: Bound B on the bit length of the lacunary-shifted representation

1. Choose a prime p with $p \in O(B^{O(1)})$
2. Evaluate $f(0), f(1), \ldots, f(p-1) \operatorname{rem} p$ to interpolate $f^{(p)}$.
3. Use a dense sparsest shift method to compute α_p
4. Repeat Steps 1–3 enough times to recover α

Example

Unknown Polynomial in $\mathbb{Q}[x]$

$f = (x-3)^{107} - 485(x-3)^{54}$

1. Choose a prime p with $p \in O(B^{O(1)})$

Step 1

$$p = 11$$

Example

Unknown Polynomial in $\mathbb{Q}[x]$

$f = (x-3)^{107} - 485(x-3)^{54}$

1. Choose a prime p with $p \in O(B^{O(1)})$
2. Evaluate $f(0), f(1), \ldots, f(p-1) \operatorname{rem} p$ to interpolate $f^{(p)}$

Step 2

$$f(0), f(1), \ldots, f(p-1) \operatorname{rem} p = 9, 10, 9, 0, 0, 2, 5, 2, 5, 10, 3$$

$$f^{(p)} = x^7 + x^6 + 2x^5 + 9x^3 + 2x^2 + 8x + 9$$

Example

Unknown Polynomial in $\mathbb{Q}[x]$

$f = (x-3)^{107} - 485(x-3)^{54}$

1. Choose a prime p with $p \in O(B^{O(1)})$
2. Evaluate $f(0), f(1), \ldots, f(p-1) \operatorname{rem} p$ to interpolate $f^{(p)}$
3. Use a dense sparsest shift method to compute α_p

Step 3

$$f^{(p)} \equiv (x-3)^7 + 10(x-3)^4 \mod p$$

$$\alpha_p = 3$$

Example

Unknown Polynomial in $\mathbb{Q}[x]$

$f = (x-3)^{107} - 485(x-3)^{54}$

1. Choose a prime p with $p \in O(B^{O(1)})$
2. Evaluate $f(0), f(1), \ldots, f(p-1) \operatorname{rem} p$ to interpolate $f^{(p)}$
3. Use a dense sparsest shift method to compute α_p
4. Repeat Steps 1–3 enough times to recover α

Step 4

$$\alpha_{11} = 3, \quad \alpha_{13} = 3, \quad \alpha_{17} = 3, \quad \ldots$$

$$\alpha = 3$$

When Can Failures Occur?

- Every step is guaranteed to succeed, except when α_p is not the unique sparsest shift of $f^{(p)}$.

Theorem

The method succeeds whenever $\deg f^{(p)} \geq 2t$.

Next we develop sufficient conditions on p to avoid failure.

Exponents Too Small

Sparsest shift of $f^{(p)}$ is not α_p

$$f = -4(x-2)^{145} + 14(x-2)^{26} + 3$$

$$p = 13$$

$$\begin{aligned} f^{(13)} &= 9(x-2)^1 + (x-2)^2 + 3 \\ &\equiv (x-4)^2 + 12 \end{aligned}$$

Condition: $(p-1) \nmid e_t(e_t-1)(e_t-2)\cdots(e_t-(2t-2))$

Exponents Collide

Sparsest shift of $f^{(p)}$ is not α_p

$$f = 4(x-1)^{59} + 2(x-1)^{21} + 7(x-1)^{19} + 20$$

$$p = 11$$

$$\begin{aligned} f^{(11)} &= 4(x-1)^9 + 2(x-1)^1 + 7(x-1)^9 + 9 \\ &= 2(x-1) + 9 \\ &\equiv 2(x-2) \end{aligned}$$

Condition: $(p-1) \nmid (e_t - e_1)(e_t - e_2)\cdots(e_t - e_{t-1})$

Coefficients Vanish

Sparsest shift of $f^{(p)}$ is not α_p

$$f = 69(x-5)^{42} - 12(x-5)^{23} + 4$$

$$p = 23$$

$$\begin{aligned} f^{(23)} &= 0(x-5)^{20} + 11(x-5)^{1} + 4 \\ &= 11(x-5) + 4 \\ &\equiv 11(x-13) \end{aligned}$$

Condition: $p \nmid c_t$

Sufficient Conditions

Definition

$$C = \max\{e_1, 1\} \cdot \prod_{i=2}^{t-1} e_i \cdot \prod_{i=1}^{t-1} (e_t - e_i) \cdot \prod_{i=0}^{2t-2} (e_t - i) \quad \leq 2^{4B^2}$$

Sufficient Conditions for Success

- $p \nmid c_t$
- $(p-1) \nmid C$

Approach

Choose primes $p = qk + 1$, for distinct primes q.
$q \mid (p-1)$, so $q \nmid C \Rightarrow (p-1) \nmid C$.

Generating Good Primes

Definition

For $q \in \mathbb{Z}$, $S(q)$ is the smallest prime p such that $q|(p-1)$.

Theorem (Mikawa 2001)

There exists a constant μ such that, for all $n > \mu$, and for most of the integers $q \in \{n, n+1, \ldots, 2n\}$, $S(q) < q^{1.89}$.

Prime Choosing Algorithm

1. $q \leftarrow$ next smallest prime
2. If $S(q) < q^{1.89}$, add $S(q)$ to $\mathcal{P}$
3. Repeat until $|P|$ is sufficiently large.

$\mathcal{P}$ must contain $\Omega(B^2)$ "good" primes,
so we only iterate $\tilde{O}(B^2)$ times.

Complexity

- Step 3 (computing sparsest shift of $f^{(p)}$) dominates.
- Cost of this step is $O(B^2 \mathrm{M}(B^3 p^2))$ bit operations (using deterministic algorithm from Giesbrecht et al. 2003)

Bit Complexity

$$O(B^3 \cdot \mathrm{M}(B^{10.56} \log^{15.12} B) \cdot \mathrm{M}(\log^2 B) = \tilde{O}(B^{13.56})$$

Can be faster by using probabilistic methods and/or heuristics.

Problem

Given $\alpha \in \mathbb{Q}$, we have a modular black box for the t-sparse polynomial $f(x+\alpha) \in \mathbb{Q}[x]$.

Question

How to recover $f \in \mathbb{Q}[x]$ from $\alpha \in \mathbb{Q}$ and images $f^{(p)} \in \mathbb{Z}_p[x]$?

Example

$$\begin{aligned} f &= 12x^{46} + 3x^{29} + 9x^{1}2 \\ f^{(7)} &= 3x^{6} + 3x^{5} + 5x^{4} \\ f^{(11)} &= 3x^{9} + x^{6} + 9x^{2} \end{aligned}$$

How can we match up the exponents?

Summary of Sparsest Shift Computation Techniques

Deterministic Algorithm

Actual complexity only $\tilde{O}(B^{29})$ unless ERH is false.

Probabilistic Algorithm

Always correct and (provably) probably much faster.

Heuristic

Faster in practice and provably never wrong, but might not terminate

Interpolation

Once α is known, we can construct a modular black box for evaluating $f(x + \alpha)$.

Then use lacunary interpolation along the lines of Kaltofen, Lakshman, & Wiley (1990) and Avendaño, Krick, & Pacetti (2006).

Conclusions

- Shifted-lacunary interpolation can be performed in polynomial time, for rational polynomials given by a modular black box.
- How to apply these techniques to other problems on lacunary polynomials?
- What about domains other than $\mathbb{Q}[x]$?
- What about multivariate rational polynomials?