Sparse interpolation and small primes in arithmetic progressions

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Interpolating an unknown polynomial

Example

$$f = (x-3)^{107} - 485(x-3)^{54}$$

Suppose we can evaluate $f(\theta)$ at any chosen point θ.

- Can we find a formula for f?

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$$f = (x-3)^{107} - 485(x-3)^{54}$$

Suppose we can evaluate $f(\theta)$ at any chosen point θ.

- Can we find a simple formula for f
  in a reasonable amount of time?

Shifted-Lacunary Interpolation

Want to interpolate an unknown polynomial $f \in \mathbb{Q}[x]$ into:

Definition (Shifted-Lacunary Representation)

$$f = c_0 + c_1(x-\alpha)^{e_1} + c_2(x-\alpha)^{e_2} + \cdots + c_t(x-\alpha)^{e_t},$$

where $e_1 < \cdots < e_t = n$ and t is minimal for any α

- Can be reduced to finding the sparsest shift α.
- No previous polynomial-time algorithm known.

Black Box Model

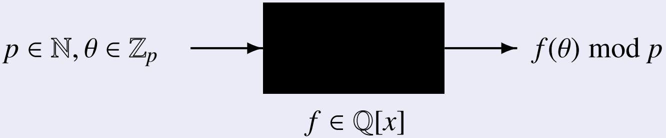
Arbitrary evaluations will usually be very large:

Example

$$
\begin{aligned}
f &= (x-3)^{107} - 485(x-3)^{54} \\
f(1) &= -162259276829222100374855109050368
\end{aligned}
$$

To control evaluation size, use modular arithmetic:

The "Modular Black-Box"



Modular-Reduced Polynomial

Definition (Modular-reduced Polynomial)

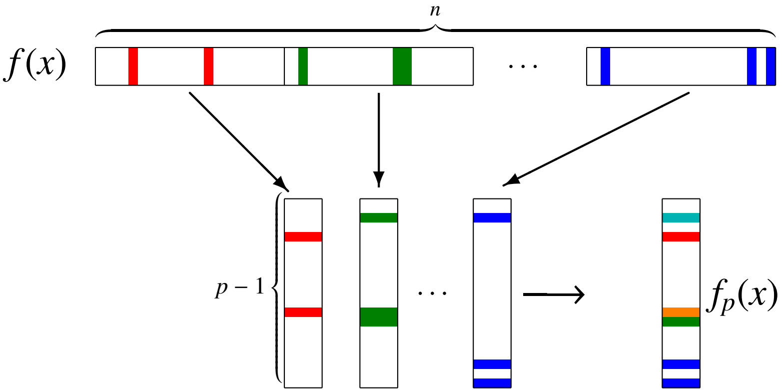
For $f \in \mathbb{Q}[x]$, $f^{(p)}$ is the unique polynomial in $\mathbb{Z}_p[x]$ with degree less than p such that $f \equiv f^{(p)} \bmod (x^p - x)$.

- $f(\theta) \operatorname{rem} p = f^{(p)}(\theta \operatorname{rem} p), \qquad \forall \theta \in \mathbb{Z}$ (Fermat's Little Theorem)
- α_p, the sparsest shift of $f^{(p)}$, is at worst a t-sparse shift of $f^{(p)}$

Example

$$
\begin{aligned}
f &= -4(x-2)^{145} + 14(x-2)^{26} + 3 \\
f^{(11)} &= 3x^6 + 4x^5 + 9x^3 + 6x^2 + 6x + 4 \\
&= 7(x-2)^5 + 3(x-2)^6 + 3
\end{aligned}
$$

Visual description of $f^{(p)}$



Uniqueness and Rationality of Sparsest Shift

Theorem (Lakshman & Saunders (1996))

If the degree is at least twice the sparsity,
then the sparsest shift is unique and rational.

Corollary

If $\deg f^{(p)} \geq 2t$, then $\alpha_p \equiv \alpha \pmod{p}$

Condition not satisfied means p is a “bad prime”,
or the polynomial is dense.

Outline of Algorithm

Input: Modular black box for $f \in \mathbb{Q}[x]$,
Bound B on the bit length of the lacunary-shifted representation

1. Choose a prime p
2. Evaluate $f(0), f(1), \ldots, f(p-1) \operatorname{rem} p$ to interpolate $f^{(p)}$.
3. If $\deg f^{(p)} \geq 2t$, then compute sparsest shift α_p of $f^{(p)}$
4. Repeat Steps 1–3 enough times to recover α
5. Use sparse interpolation to recover $f(x+\alpha)$

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The challenge: Choosing primes on Step 1
so that Step 3 will succeed

Bad prime: Exponents Too Small

Sparsest shift of $f^{(p)}$ is not α_p

$$f = -4(x-2)^{145} + 14(x-2)^{26} + 3$$

$$p = 13$$

$$\begin{aligned} f^{(13)} &= 9(x-2)^{1} + (x-2)^{2} + 3 \\ &\equiv (x-4)^{2} + 12 \end{aligned}$$

Condition: $(p-1) \nmid e_t(e_t-1)(e_t-2)\cdots(e_t-(2t-2))$

Bad prime: Exponents Collide

Sparsest shift of $f^{(p)}$ is not α_p

$$f = 4(x-1)^{59} + 2(x-1)^{21} + 7(x-1)^{19} + 20$$

$$p = 11$$

$$\begin{aligned} f^{(11)} &= 4(x-1)^{9} + 2(x-1)^{1} + 7(x-1)^{9} + 9 \\ &= 2(x-1) + 9 \\ &\equiv 2(x-2) \end{aligned}$$

Condition: $(p-1) \nmid (e_t - e_1)(e_t - e_2)\cdots(e_t - e_{t-1})$

Bad prime: Coefficients Vanish

Sparsest shift of $f^{(p)}$ is not α_p

$$f = 69(x-5)^{42} - 12(x-5)^{23} + 4$$

$$p = 23$$

$$\begin{aligned} f^{(23)} &= 0(x-5)^{20} + 11(x-5)^{1} + 4 \\ &= 11(x-5) + 4 \\ &\equiv 11(x-13) \end{aligned}$$

Condition: $p \nmid c_t$

Sufficient Conditions

Definition

$$C = \prod_{i=1}^{t-1} e_i \cdot \prod_{i=1}^{t-1} (e_t - e_i) \cdot \prod_{i=0}^{2t-2} (e_t - i) \quad \leq 2^{4B^2}$$

Sufficient Conditions for Success

- $p \nmid c_t$
- $(p-1) \nmid C$

Generating Good Primes

The Problem:

Given $\beta_1 > \log_2 c_t$, $\beta_2 > \log_2 C$, and ℓ
find ℓ small primes p such that $p \nmid c_t$ and $(p-1) \nmid C$.

Primes-in-arithmetic-progressions approach

Choose primes $p = qk + 1$, for distinct primes q.
$q \mid (p-1)$, so $q \nmid C \Rightarrow (p-1) \nmid C$.

A brief history of primes

(in arithmetic progressions)

Definition

For $q \in \mathbb{Z}$, $S(q)$ is the smallest prime p such that $q|(p-1)$.

- Dirichlet (1837): $S(q)$ exists
- Linnik (1944): $S(q) < cq^L$ for some $L > 0$
- Heath-Brown (1992): $S(q) < cq^{5.5}$

For most q:

- Bombieri, Friedlander, Iwaniec (1987): $S(q) < cq^2$
- Rousselet (1988): $S(q) < q^2$ (more explicit)
- Baker & Harman (1996): $S(q) < q^{1.93}$
- Mikawa (2001): $S(q) < q^{1.89}$

Mikawa's Result

Fact (Mikawa 2001)

There exists a constant μ such that, for all $n > \mu$, and for most of the integers $q \in \{n, n+1, \ldots, 2n\}$, with less than $\mu n / \log^2 n$ exceptions, $S(q) < q^{1.89}$.

Theorem

For any $k \in \mathbb{N}$, we can construct a set $\mathcal{Q}$ of primes such that the set $\mathcal{P} = \{S(q) : q \in \mathcal{Q}\}$ has at least k distinct elements, and each $p \in \mathcal{P}$ is $O(k^{1.89} \cdot \log^{1.89} k)$.

Proof of Prime Generation Theorem

Proof sketch

For convenience, define $\Upsilon(n) = \dfrac{3n}{5\log n} - \dfrac{\mu n}{\log^2 n}$

Let $n \in \mathbb{N}$ such that $n > 21$, $n > \mu$, and $\Upsilon(n) > k$. (μ is just a guess.)

Define $\mathcal{Q} = \left\{q \text{ prime: } n \le q < 2n \text{ and } S(q) < q^{1.89}\right\}$

Since $n > 21$, $\#\left\{q \text{ prime: } n \le q < 2n\right\} \ge 3n/(5\log n)$.

Therefore $\#\mathcal{P} = \#\mathcal{Q} > \Upsilon(n) > k$.

(If not, then μ was too small, so double it.)

To make $\Upsilon(n) > k$, we have $n \in O(k \log k)$.
Each $p \in \mathcal{P}$ is $O(n^{1.89})$, so $p \in O(k^{1.89} \cdot \log^{1.89} k)$.

Complexity Implications

Theorem

Given a modular black box for unknown $f \in \mathbb{Q}[x]$ *and a bound* B *on the bit-length of the shifted-lacunary representation, we can compute the sparsest shift* α *with* $O(B^{8.56} \log^{6.78} \cdot (\log\log B)^2 \cdot \log\log\log B)$ *bit operations.*

In fact, we have confirmed for all word-sized q that $S(q) < 2q \log^2 q$.
Proving this would improve the complexity to $\tilde{O}(B^5)$.

A different idea

We have chosen $\mathcal{P}$ so that the following is sufficiently large:

$$\operatorname{lcm}\left(\{p-1 : p \in \mathcal{P}\}\right) \geq \prod_{q \in Q} q$$

How about using this directly?

- Use the first n primes for the algorithm: $p_1, p_2, \ldots, p_n$.
- Define $\Psi(n) = \operatorname{lcm}(p_1 - 1,\ p_2 - 1,\ \ldots,\ p_n - 1)$.
- If $\Psi(n) > C \cdot p_n^k$, then at least k of the first n primes satisfy $(p_i - 1) \nmid C$.

Lower bounds on $\Psi(n)$

Notice that $\Psi(n) = \lambda(P_n)$, where
$\lambda(x)$ is the Carmichael lambda function,
and $P_n = \prod_{i=1}^{n} p_i$ is the nth primorial number.

There are lower bounds on $\lambda(x)$, but they are either
too weak to be useful, or have too many exceptions.

Experimentally, we see strong evidence that $\Psi(n) \gg 2^n$.
This would also give a $\tilde{O}(B^5)$ complexity
(with smaller log factors)

Summary

- Shifted-lacunary interpolation can be performed in polynomial time, for rational polynomials given by a modular black box.
- How to apply these techniques to other problems on lacunary polynomials?
- What about domains other than $\mathbb{Q}[x]$?
- Prove that $S(q) \ll q \log^2 q$.
- Prove that $\Psi(n) \ll 2^n$.