Fast Multiplication with Low Space Complexity

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A Fun Puzzle

Why care about space complexity?

- Physical restrictions on space; not on time
- Cache misses incur a significant penalty
  in modern architectures
- Specific applications (e.g. embedded devices)
- Theoretical interest

Multiplication Algorithms

(over $\mathbb{Z}$ or $R[x]$)

	Time Complexity	**Space Complexity**
Classical Method	$O(n^2)$	$O(1)$
Divide-and-Conquer Karatsuba/Ofman '63	$O(n^{\log_2 3})$ or $O(n^{1.59})$	$O(n)$
FFT-based Schönhage/Strassen '71 Cantor/Kaltofen '91	$O(n \log n \log \log n)$	$O(n)$

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Time-Space Tradeoff: Product of time and space is $\Omega(n^2)$
(Savage & Swamy 1979; Abrahamson 1986)

Standard Space Complexity Model (Papadimitriou)

3-Tape Turing Machine:

- Input tape (read-only)

9	1	1	2	$\times$	7	2	6

- Work tape (read/write)
  Size of this tape determines space complexity

4	$\cdot$	2	$\cdot$	1	8

- Output tape (write only)

$\cdots$	5	3	1	2

Significant improvements not possible in this model

Our Space Complexity Model

3-Tape Turing Machine:

- Input tape (read-only)

9	1	1	2	$\times$	7	2	6

- Work tape (read/write)
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$\cdots$	5	3	1	2

More realistic model for modern computers

Previous Work

- Monagan 1993: Importance of space efficiency for multiplication over $\mathbb{Z}_p[x]$
- Maeder 1993: Bounds extra space for Karatsuba multiplication so that storage can be preallocated — about $2n$ extra memory cells required.
- Thomé 2002: Karatsuba multiplication for polynomials using n extra memory cells.
- Zimmerman & Brent 2008:
  "The efficiency of an implementation of Karatsuba's algorithm depends heavily on memory usage."

Our Contributions

	Time Complexity	Space Complexity
Classical Method	$O(n^2)$	$O(1)$
Divide-and-Conquer Karatsuba/Ofman '63	$O(n^{\log_2 3})$ or $O(n^{1.59})$	$O(\log n)$
FFT-based Schönhage/Strassen '71 Cantor/Kaltofen '91	$O(n \log n \log \log n)$	$O(2^{\lceil \log_2 n \rceil} - n)$ ($O(1)$ if $n = 2^k$)

Standard Karatsuba Algorithm

Initial Setup

Idea: Reduce one degree-$2k$ multiplication to three of degree k.

Input: $f, g \in \mathrm{R}[x]$ each with degree less than $2k$.

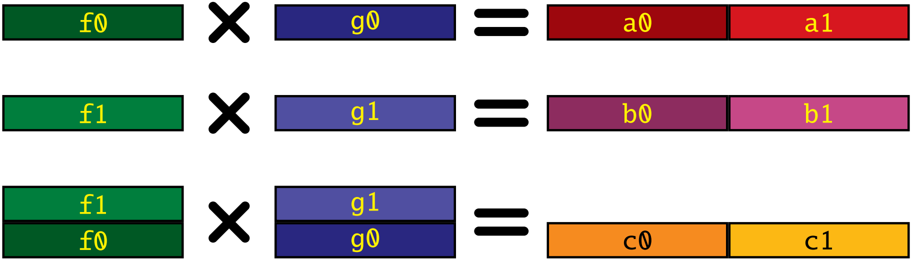
Write $f = f_0 + f_1 x^k$ and $g = g_0 + g_1 x^k$.

Standard Karatsuba Algorithm

Recursive Multiplications

Compute two sums: $f_0 + f_1$ and $g_0 + g_1$,
and three intermediate products:

$$a = f_0 \cdot g_0 \qquad b = f_1 \cdot g_1 \qquad c = (f_0 + f_1) \cdot (g_0 + g_1)$$



Standard Karatsuba Algorithm

Final Additions and Subtractions

Combine the computed products as follows:

$$
\begin{aligned}
a + (c - a - b) \cdot x^k + b \cdot x^{2k} \\
&= f_0 g_0 + (f_0 g_1 + f_1 g_0) \cdot x^k + f_1 g_1 \cdot x^{2k} \\
&= f \cdot g
\end{aligned}
$$

Extra Requirements for Improved Karatsuba

Read-Only Input Space:

f01	f11		g0	g1

Read/Write Output Space:

(empty)	(empty)	(empty)	(empty)

To Compute: $f \cdot g$

Extra Requirements for Improved Karatsuba

- The low-order coefficients of the output are initialized as h, and the product $f \cdot g$ is added to this.

Read-Only Input Space:

f01	f11		g0	g1

Read/Write Output Space:

h0	h1	(empty)	(empty)

To Compute: $f \cdot g + h$

Extra Requirements for Improved Karatsuba

- The low-order coefficients of the output are initialized as h, and the product $f \cdot g$ is added to this.
- The first polynomial f is given as a sum $f^{(0)} + f^{(1)}$.

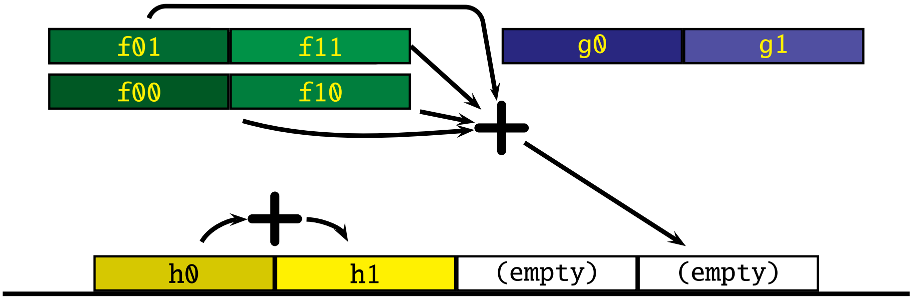
Read-Only Input Space:

Read/Write Output Space:

To Compute: $(f^{(0)} + f^{(1)}) \cdot g + h$

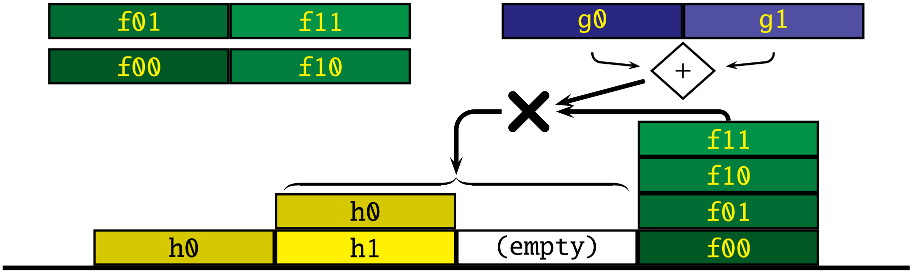
Space-Efficient Karatsuba Algorithm

Step 1: Preparing to Multiply



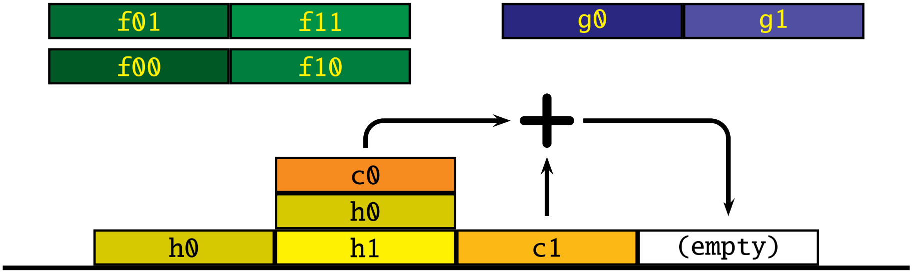
Space-Efficient Karatsuba Algorithm

Step 2: First product c



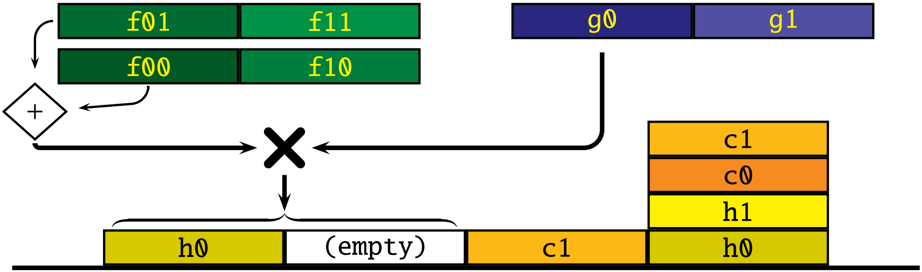
Space-Efficient Karatsuba Algorithm

Step 3: Rearranging



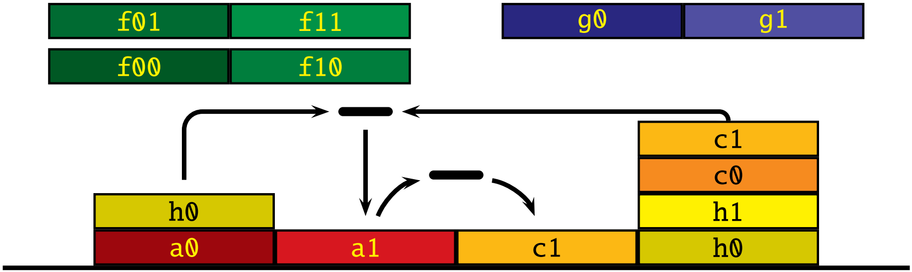
Space-Efficient Karatsuba Algorithm

Step 4: Second product a



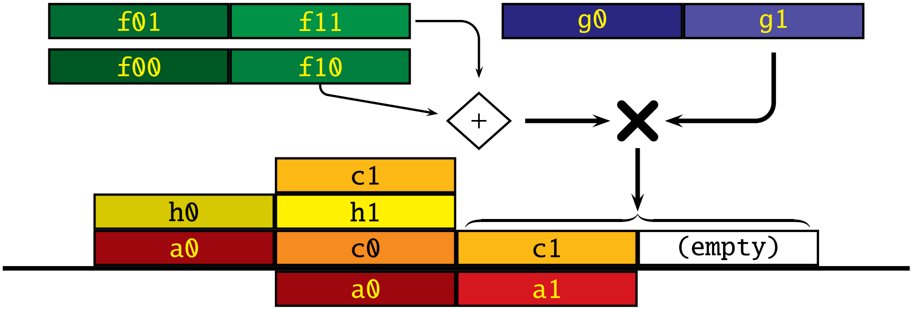
Space-Efficient Karatsuba Algorithm

Step 5: Rearranging



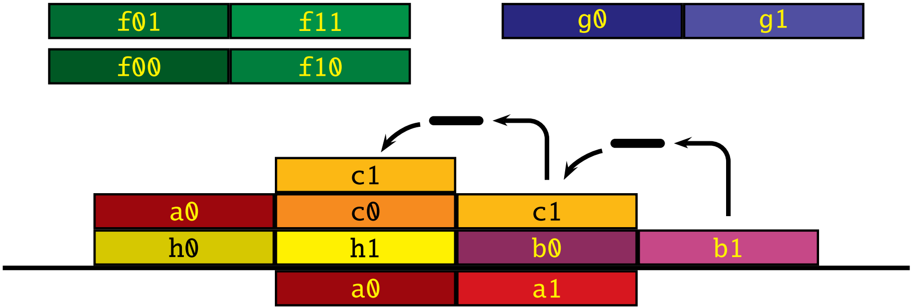
Space-Efficient Karatsuba Algorithm

Step 6: Third product b



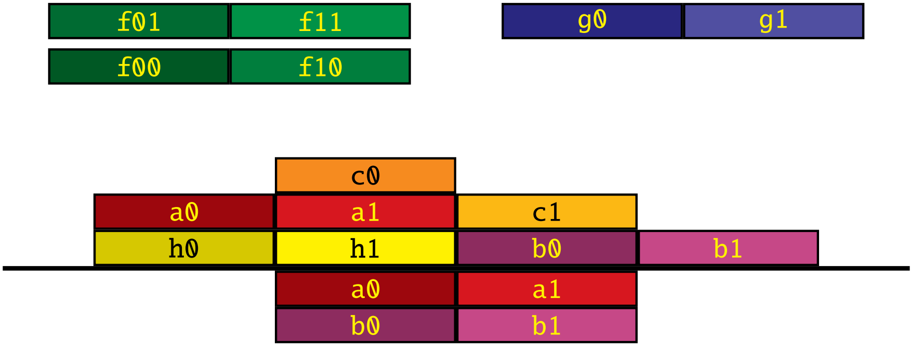
Space-Efficient Karatsuba Algorithm

Step 7: Rearranging



Space-Efficient Karatsuba Algorithm

Final Result



Analysis

- 3 recursive calls on degree-k arguments
  $\Rightarrow O(n^{\log_2 3})$ time complexity
- Constant extra space required at each recursive step
  $\Rightarrow O(\log n)$ space complexity
- At most $9n/2$ additions at each recursive step
  (compared to $4n$ for naïve implementation)

First multiplication algorithm with $o(n^2)$ time $\times$ space

Initial Recursive Calls

Call the algorithm discussed above Algorithm A

Algorithm B

- Neither operand is given as a sum
- $7n/2$ additions
- 2 recursive calls to B and one to A

Algorithm C

- Neither operand is given as a sum, and output is uninitialized
- $7n/2$ additions
- 2 recursive calls to C and one to B

Algorithm C is the top-level call.

Implementation Details

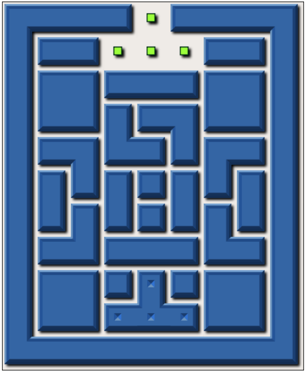
With some slight modifications, we can handle:

1. Odd-length operands
2. Different-length operands (Standard blocking method is used)

Proof-of-concept implementation in NTL

- $\approx 40\%$ slower than NTL Karatsuba
- Versions which destroy the input only $\approx 5\%$ slower
- NTL only allocates space once — not thread-safe!
- Victor Shoup is a better programmer than me

Open Problems



- More efficient implementation for univariate polynomials
- Implementation over $\mathbb{Z}$ (GMP)
- Similar results for Toom-Cook 3-way or k-way
- Better results for FFT-based multiplication
- Is completely in-place (overwriting input) possible?