Sorting

Sorting Problem

Input: An array of *comparable* elements
Output: The same elements, sorted in ascending order

- One of the most well-studied algorithmic problems
- Has lots of practical applications
- You should already know a few algorithms...

SelectionSort

```
def selectionSort(A):
    for i in range(0, len(A)-1):
        m = i
        for j in range(i+1, len(A)):
            if A[j] < A[m]:
                m = j
        swap(A, i, m)
```


InsertionSort

```
def insertionSort(A):
    for i in range(1, len(A)):
        j = i - 1
        while j >= 0 and A[j] > A[j+1]:
            swap(A, j, j+1)
            j = j - 1
```

Common Features

It's useful to look for larger patterns in **algorithm design**.

Both InsertionSort and SelectionSort build up a sorted array one element at a time, in the following two steps:

- **Pick**: Pick an element in the unsorted part of the array
- **Place**: Insert that element into the sorted part of the array

For both algorithms, one of these is "easy" (constant time) and the other is "hard" ($O(n)$ time). Which ones?

Analysis of SelectionSort

Each loop has $O(n)$ iterations, so the total cost is $O(n^2)$.

What about a big-Θ bound?

Arithmetic Series

An *arithmetic series* is one where consecutive terms differ by a constant.

General formula: $\displaystyle\sum_{i=0}^{m}(a+bi) = \frac{(m+1)(2a+bm)}{2}$

So the worst-case of SelectionSort is

This is $\Theta(n^2)$, or **quadratic time**.

Worst-Case Family

Why can't we analyze InsertionSort in the same way?

We need a **family of examples**, of arbitrarily large size, that demonstrate the worst case.

Worst-case for InsertionSort:

Worst-case cost:

SelectionSort (Recursive Version)

```
def selectionSortRec(A, start=0):
    if (start < len(A) - 1):
        m = minIndex(A, start)
        swap(A, start, m)
        selectionSortRec(A, start + 1)
```

minIndex

```
def minIndex(A, start=0):
    if start >= len(A) - 1:
        return start
    else:
        m = minIndex(A, start+1)
        if A[start] < A[m]:
            return start
        else:
            return m
```


Analysis of minIndex

Let $T(n)$ be the worst-case number of operations for a size-n input array.

We need a **recurrence relation** to define $T(n)$:

$$T(n) = \begin{cases} 1, & n \leq 1 \\ 4 + T(n-1), & n \geq 2 \end{cases}$$

Solving the recurrence:

Analysis of recursive SelectionSort

Let $S(n)$ be the worst-case for SelectionSort

What is the recurrence?

Divide and Conquer

A new **Algorithm Design Paradigm**: Divide and Conquer

Works in three steps:

1. Break the problem into similar subproblems
2. Solve each of the subproblems recursively
3. Combine the results to solve the original problem.

MergeSort and BinarySearch both follow this paradigm.
(How do they approach each step?)

MergeSort

```
def mergeSort(A):
    if len(A) <= 1:
        return A
    else:
        m = len(A) // 2
        B = A[0 : m]
        C = A[m : len(A)]
        mergeSort(B)
        mergeSort(C)
        A[:] = merge(B, C)
```

Merge

```
def merge(B, C):
    A = []
    i, j = 0, 0
    while i < len(B) and j < len(C):
        if B[i] <= C[j]:
            A.append(B[i])
            i = i + 1
        else:
            A.append(C[j])
            j = j + 1
    while i < len(B):
        A.append(B[i])
        i = i + 1
    while j < len(C):
        A.append(C[j])
        j = j + 1
    return A
```


Analysis of Merge

Each `while` loop has constant cost.
So we just need the total number of iterations through every loop.

	Lower bound	Upper bound	Exact
Loop 1	$\min(a, b)$	$a + b$	
Loop 2	0	a	
Loop 3	0	b	
Total	$\min(a, b)$	$2(a + b)$	

a is the size of A and b is the size of B.

Analysis of MergeSort

Complexity of Sorting

Algorithms we have seen so far:

Sort	Worst-case cost
SelectionSort	$\Theta(n^2)$
InsertionSort	$\Theta(n^2)$
MergeSort	$\Theta(n \log n)$
HeapSort	$\Theta(n \log n)$

Million dollar question: Can we do better than $\Theta(n \log n)$?

Comparison Model

Elements in the input array can only be accessed in two ways:

- Moving them (swap, copy, etc.)
- Comparing two of them ($<$, $>$, $=$, etc.)

Every sorting algorithm we have seen uses this model.
It is a very **general** model for sorting strings or integers or floats or anything else.

What operations are *not* allowed in this model?

Permutations

How many orderings (aka *permutations*) are there of n elements?

n factorial, written $n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$.

Observation: A comparison-based sort is only sensitive to the **ordering** of A, not the actual contents.

For example, MergeSort will do the same things on
`[1,2,4,3]`, `[34,35,37,36]`, or `[10,20,200,99]`.

Logarithms

Recall some useful facts about logarithms:

- $\log_b b = 1$
- $\log_b ac = \log_b a + \log_b c$
- $\log_b a^c = c \log_b a$
- $\log_b a = (\log_c a)/(\log_c b)$

Now how about a lower bound on $\lg n!$?

Lower Bound on Sorting

1. A correct algorithm must take different actions for each of the possible input permutations.
2. The choice of actions is determined only by comparisons.
3. Each comparison has two outcomes.
4. An algorithm that performs c comparisons can only take 2^c different actions.
5. The algorithm must perform at least $\lg n!$ comparisons.

Therefore... **ANY comparison-based sort is** $\Omega(n \log n)$

Conclusions

Any sorting algorithm that only uses comparisons must take at least $\Omega(n \log n)$ steps in the worst case.

- This means that sorts like MergeSort and HeapSort couldn't be much better — they are **asymptotically optimal**.
- What if I claimed to have a $O(n)$ sorting algorithm?
  What would that tell you about my algorithm (or about me)?
- Remember what we learned about **summations**,
  recursive algorithm analysis, and **logarithms**.