Representing Big Integers

Multiple-precision integers require more storage than a single machine word like an 'int'.

Remember: why are these important computationally?

Example: `4391354067575026` represented as an array:

- In base $B = 10$:
  `[6, 2, 0, 5, 7, 5, 7, 6, 0, 4, 5, 3, 1, 9, 3, 4]`
- In base $B = 256$:
  `[242, 224, 71, 203, 233, 153, 15]`

Base of representation

General form of a multiple-precision integer:
$d_0 + d_1 B + d_2 B^2 + d_3 B^3 + \cdots + d_{n-1} B^{n-1}$,

Does the choice of base B matter?

Addition

How would you add two n-digit integers?

- Remember, every digit is in a separate machine word.
- How big can the "carries" get?
- What if the inputs don't have the same size?
- How fast is your method?

Standard Addition

```
def add(X, Y, B):
    carry = 0
    A = zero-filled array of length (len(X) + 1)
    for i in range(0, len(Y)):
        carry, A[i] = divmod(X[i] + Y[i] + carry, B)
    for i in range(len(Y), len(X)):
        carry, A[i] = divmod(X[i] + carry, B)
    A[len(X)] = carry
    return A
```


Linear-time lower bounds

Remember the $\Omega(n \log n)$ lower bound for comparison-based sorting?

Much easier lower bounds exist for many problems!

Linear lower bounds

For any problem with input size n,
where **changing any part of the input could change the answer**,
any correct algorithm must take $\Omega(n)$ time.

What does this tell us about integer addition?

Multiplication

Let's remember how we multiplied multi-digit integers in grade school.

Standard multiplication

```
def smul(X, Y, B):
    n = len(X)
    A = zero-filled array of length (2*n)
    T = zero-filled array of length n
    for i in range(0, n):
        # set T = X * Y[i]
        carry = 0
        for j in range(0, n):
            T[j] = (X[j] * Y[i] + carry) % B
            carry = (X[j] * Y[i] + carry) // B
        # add T to A, the running sum
        A[i : i+n+1] = add(A[i : i+n], T[0 : n], B)
        A[i+n] += carry
    return A
```


Divide and Conquer

Maybe a divide-and-conquer approach will yield a faster multiplication algorithm.

Let's split the digit-lists in half. Let $m = \lfloor \frac{n}{2} \rfloor$ and write $x = x_0 + B^m x_1$ and $y = y_0 + B^m y_1$.
Then we multiply $xy = x_0 y_0 + x_0 y_1 B^m + x_1 y_0 B^m + x_1 y_1 B^{2m}$.

For example, if $x = 7407$ and $y = 2915$, then we get

```
 Integers  | Array representation
-----------|---------------------
 x = 7407  |  X = [7, 0, 4, 7]
 y = 2915  |  Y = [5, 1, 9, 2]
x0 =   07  | X0 = [7, 0]
x1 =   74  | X1 = [4, 7]
y0 =   15  | Y0 = [5, 1]
y1 =   29  | Y1 = [9, 2]
```


Recurrences for Multiplication

Standard multiplication has running time

$$T(n) = \begin{cases} 1, & n = 1 \\ n + T(n-1), & n \geq 2 \end{cases}$$

The divide-and-conquer way has running time

$$T(n) = \begin{cases} 1, & n = 1 \\ n + 4T(\frac{n}{2}), & n \geq 2 \end{cases}$$

Karatsuba's Algorithm

The equation:

$$(x_0+x_1B^m)(y_0+y_1B^m) = x_0y_0+x_1y_1B^{2m}+((x_0+x_1)(y_0+y_1)-x_0y_0-x_1y_1)B^m$$

leads to an algorithm:

1. Compute two sums: $u = x_0 + x_1$ and $v = y_0 + y_1$.
2. Compute three m-digit products: x_0y_0, x_1y_1, and uv.
3. Sum them up and multiply by powers of B to get the answer:
   $xy = x_0y_0 + x_1y_1B^{2m} + (uv - x_0y_0 - x_1y_1)B^m$

Karatsuba Example

```
    x = 7407 = 7 + 74*100
    y = 2915 = 15 + 29*100

    u = x0 + x1 =  7 + 74 = 81
    v = y0 + y1 = 15 + 29 = 44

x0*y0 =  7*15 =  105
x1*x1 = 74*29 = 2146
 u*v  = 81*44 = 3564

 x*y  = 105 + 2146*10000 + (3564 - 105 - 2146)*100
      =        105
        +   1313
        + 2146
      =   21591405
```


Analyzing Karatsuba

$$T(n) = \begin{cases} 1, & n \leq 1 \\ n + 3T(\frac{n}{2}), n \geq 2 \end{cases}$$

Crucial difference: The coefficient of $T(\frac{n}{2})$.

Beyond Karatsuba

History Lesson:

- 1962: Karatsuba: $O(n^{1.59})$
- 1963: Toom & Cook: $O(n^{1.47})$, $O(n^{1+\epsilon})$
- 1971: Schönhage & Strassen: $O(n(\log n)(\log\log n))$
- 2007: Fürer: $O(n(\log n)2^{\log * n})$

Lots of work to do in algorithms!

Recurrences

Algorithm	Recurrence	Asymptotic big-Θ
BinarySearch	$1 + T(n/2)$	$\log n$
LinearSearch	$1 + T(n-1)$	n
MergeSort (space)	$n + T(n/2)$	n
MergeSort (time)	$n + 2T(n/2)$	$n \log n$
KaratsubaMul	$n + 3T(n/2)$	$n^{\lg 3}$
SelectionSort	$n + T(n-1)$	n^2
StandardMul	$n + 4T(n/2)$	n^2

Master Method A

$$T(n) = aT(\frac{n}{b}) + n^c(\log n)^d$$

Write $e = \log_b a = \frac{\lg a}{\lg b}$

Three cases:

1. $\mathbf{c = e}$. Then $T(n) \in \Theta(n^c(\log n)^{d+1})$.
2. $\mathbf{c < e}$. Then $T(n) \in \Theta(n^e) = \Theta(n^{\log_b a})$.
3. $\mathbf{c > e}$. Then $T(n) \in \Theta(n^c(\log n)^d)$.

Master Method B

$$T(n) = aT(n-b) + n^c(\log n)^d$$

Two cases:

1. $\mathbf{a = 1}$. Then $T(n) \in \Theta(n^{c+1}(\log n)^d)$.
2. **a** $>$**1**. Then $T(n) \in \Theta(e^n)$, where e is the positive constant $a^{1/b}$.

Matrix Multiplication

Review: **Dimensions** = number of rows and columns.

Multiplication of 4×3 and 4×2 matrices:

$$\begin{bmatrix} 7 & 1 & 2 \\ 6 & 2 & 8 \\ 9 & 6 & 3 \\ 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 6 & 3 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 28 & 9 \\ 56 & 30 \\ 66 & 27 \\ 24 & 15 \end{bmatrix}$$

$$A \qquad B \qquad = \qquad AB$$

Middle dimensions **must** match.

Running time:

Divide and Conquer Matrix Multiplication

$$\left[\begin{array}{c|c} S & T \\ \hline U & V \end{array}\right] \left[\begin{array}{c|c} W & X \\ \hline Y & Z \end{array}\right] = \left[\begin{array}{c|c} SW+TY & SX+TZ \\ \hline UW+VY & UX+VZ \end{array}\right]$$

Is this faster?

Strassen's Algorithm

Step 1: Seven products

$$
\begin{array}{ll}
P_1 = S(X-Z) & P_5 = (S+V)(W+Z) \\
P_2 = (S+T)Z & P_6 = (T-V)(Y+Z) \\
P_3 = (U+V)W & P_7 = (S-U)(W+X) \\
P_4 = V(Y-W) &
\end{array}
$$

Step 2: Add and subtract

$$
\left[\begin{array}{c|c} S & T \\ \hline U & V \end{array}\right]
\left[\begin{array}{c|c} W & X \\ \hline Y & Z \end{array}\right]
=
\left[\begin{array}{c|c} P_5+P_4-P_2+P_6 & P_1+P_2 \\ \hline P_3+P_4 & P_1+P_5-P_3-P_7 \end{array}\right]
$$

Fibonacci

Here's a basic algorithm to compute f_n:

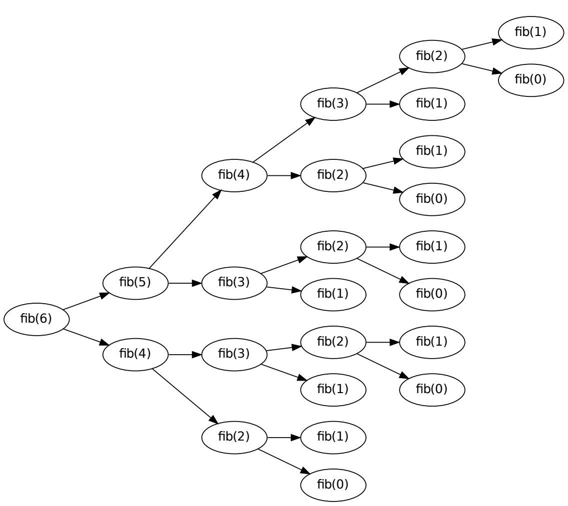
`fib(n)`

Input: Non-negative integer n
Output: f_n

```
def fib(n):
    if n <= 1:
        return n
    else:
        return fib(n-1) + fib(n-2)
```

Is this fast?

Recursion tree for `fib(6)`



Memoization

How to avoid **repeated, identical function calls**?

Memoization means saving the results of calls in a table:

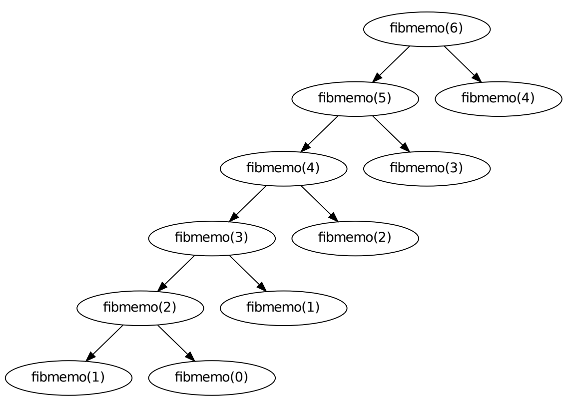
`fibmemo(n)`

Input: Non-negative integer n
Output: f_n

```
fib_table = {} # empty hash table
def fib_memo(n):
    if n not in fib_table:
        if n <= 1:
            return n
        else:
            fib_table[n] = fib_memo(n-1) + fib_memo(n-2)
    return fib_table[n]
```

See the original function?

Recursion tree for `fibmemo(6)`



Cost of Memoization

- How should the table `T` be implemented?
- Analysis

Matrix Chain Multiplication

Problem

Given n matrices $A_1, A_2, \ldots, A_n$, find the best **order of operations** to compute the product $A_1 A_2 \cdots A_n$.

Matrix multiplication is associative but *not* commutative.

In summary: where should we put the parentheses?

Example

$$\underset{\substack{X \\ 5\times 2}}{\begin{bmatrix} 4 & 9 \\ 1 & 6 \\ 9 & 7 \\ 0 & 9 \\ 2 & 0 \end{bmatrix}} * \underset{\substack{Y \\ 2\times 6}}{\begin{bmatrix} 2 & 1 & 5 & 6 & 4 & 5 \\ 8 & 0 & 9 & 1 & 8 & 4 \end{bmatrix}} * \underset{\substack{Z \\ 6\times 3}}{\begin{bmatrix} 6 & 5 & 4 \\ 8 & 8 & 5 \\ 4 & 4 & 4 \\ 0 & 7 & 0 \\ 6 & 4 & 2 \\ 1 & 7 & 5 \end{bmatrix}}$$

Computing minimal mults

Idea: Figure out the *final* multiplication, then use recursion to do the rest.

`mm(D)`

Input: Dimensions array `D` of length $n+1$
Output: Least number of mults to compute the matrix chain product

```
def mm(D):
    n = len(D) - 1
    if n == 1:
        return 0
    else:
        fewest = float('inf') # (just a placeholder)
        for i in range(1, n):
            t = (   mm(D[0 : i+1])
                 + D[0]*D[i]*D[n]
                 + mm(D[i : n+1]) )
            if t < fewest:
                fewest = t
        return fewest
```

Analyzing mm(D)

$$T(n) = \begin{cases} 1, & n = 1 \\ n + \sum_{i=1}^{n-1}(T(i) + T(n-i)), & n \geq 2 \end{cases}$$

Memoized minimal mults

Let's use our **general tool** for avoiding repeated recursive calls:

mmm(D)

Input: Dimensions array D of length $n+1$
Output: Least number of mults to compute the matrix chain product

```
mm_table = {}
def mmm(D):
    n = len(D) - 1
    if D not in mm_table:
        if n == 1: mm_table[D] = 0
        else:
            fewest = float('inf')
            for i in range(1, n):
                t = (   mmm(D[0 : i+1])
                      + D[0]*D[i]*D[n]
                      + mmm(D[i : n+1])  )
                if t < fewest: fewest = t
            mm_table[D] = fewest
    return mm_table[D]
```


Analyzing mmm(D)

- Cost of each call, not counting recursion:

- Total number of recursive calls:

Problems with Memoization

1. What data structure should T be?
2. Tricky analysis
3. Too much memory?

Solution: Dynamic Programming

- Store the table T explicitly, for a **single problem**
- Fill in the entries of T needed to solve the current problem
- Entries are computed **in order** so recursion is never required
- Final answer can be looked up in the filled-in table

Dynamic Minimal Mults Example

Multiply (8×5) times (5×3) times (3×4) times (4×1) matrices.

$D = [8, 5, 3, 4, 1]$, $n = 4$

Make a table for the value of mm(D[i..j]):

	0	1	2	3	4
0					
1					
2					
3					
4					

Dynamic Minimal Mults Algorithm

dmm(D)

```
def dmm(*D):
    n = len(D) - 1
    A = new (n+1) by (n+1) array
    for diag in range(1,n+1):
        for row in range(0, n-diag+1):
            col = diag + row
            # This part is just like the original!
            if diag == 1:
                A[row][col] = 0
            else:
                A[row][col] = float('inf')
                for i in range(row+1, col):
                    t = (   A[row][i]
                          + D[row]*D[i]*D[col]
                          + A[i][col]          )
                    if t < A[row][col]:
                        A[row][col] = t
    return A[0][n]
```