Module 4: Dictionaries and Balanced Search Trees

CS 240 - Data Structures and Data Management

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Dictionary ADT

A *dictionary* is a collection of *items*,
each of which contains a *key* and some *data*
and is called a *key-value pair* (KVP).
Keys can be compared and are typically unique.

Operations:

- *search*(k)
- *insert*(k, v)
- *delete*(k)
- optional: *join*, *isEmpty*, *size*, *etc.*

Examples: symbol table, license plate database

Elementary Implementations

Common assumptions:

- Dictionary has n KVPs
- Each KVP uses constant space
  (if not, the "value" could be a pointer)
- Comparing keys takes constant time

Unordered array or linked list

search $\Theta(n)$

insert $\Theta(1)$

delete $\Theta(1)$ (after a search)

Ordered array or linked list

search $\Theta(\log n)$

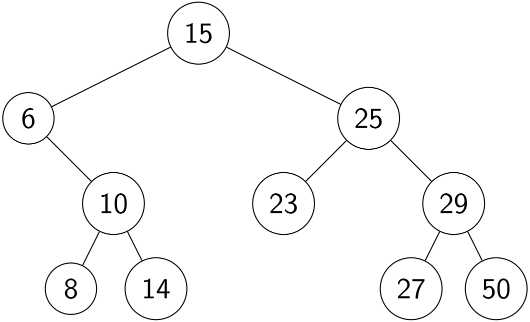
insert $\Theta(n)$

delete $\Theta(n)$

Binary Search Trees (review)

Structure A BST is either empty or contains a KVP, left child BST, and right child BST.

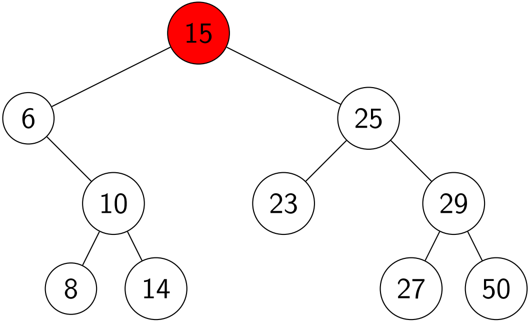
Ordering Every key k in $T.left$ is less than the root key.
Every key k in $T.right$ is greater than the root key.



BST Search and Insert

search(k) Compare k to current node, stop if found, else recurse on subtree unless it's empty

Example: *search*(24)



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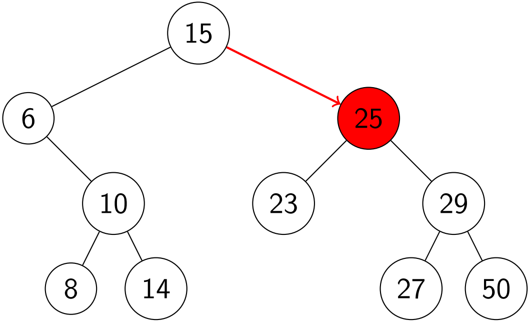

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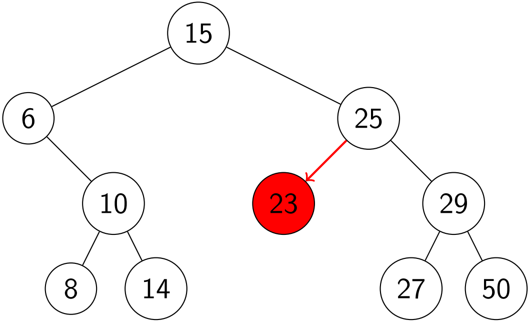

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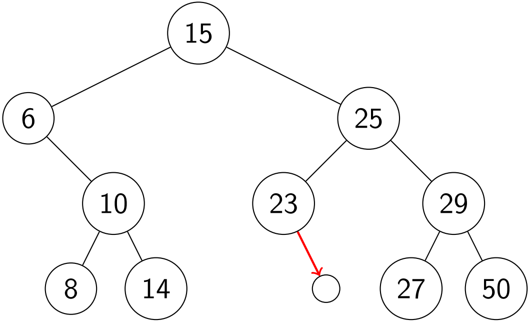
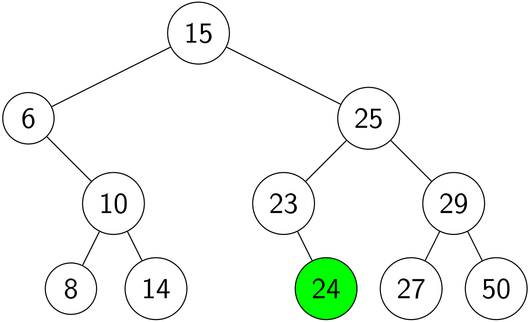


BST Search and Insert

search(k) Compare k to current node, stop if found, else recurse on subtree unless it's empty

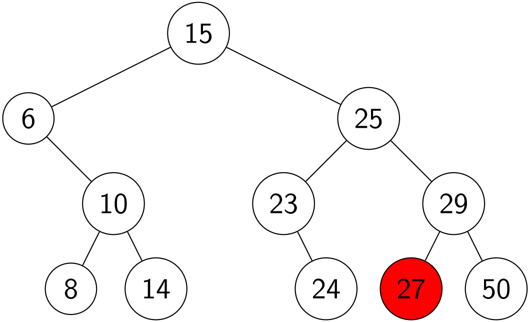
insert(k, v) Search for k, then insert (k, v) as new node

Example: *insert*(24, . . .)



BST Delete

- If node is a leaf, just delete it.



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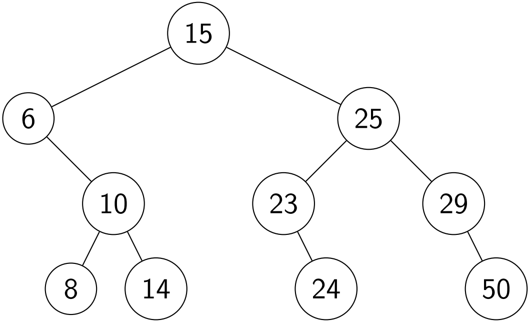
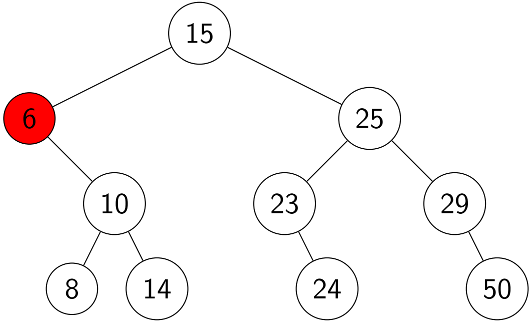


BST Delete

- If node is a leaf, just delete it.
- If node has one child, move child up



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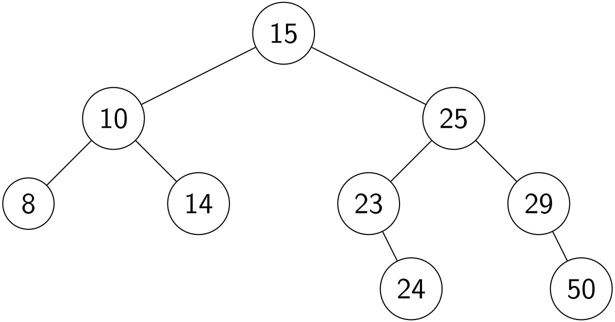

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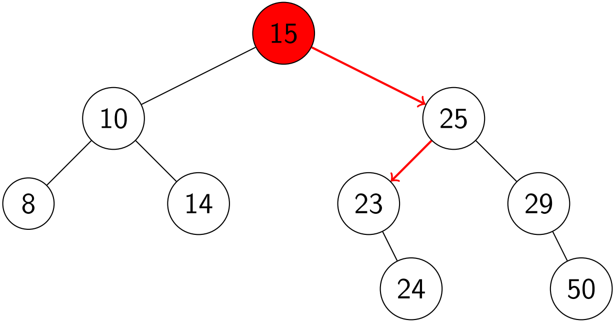

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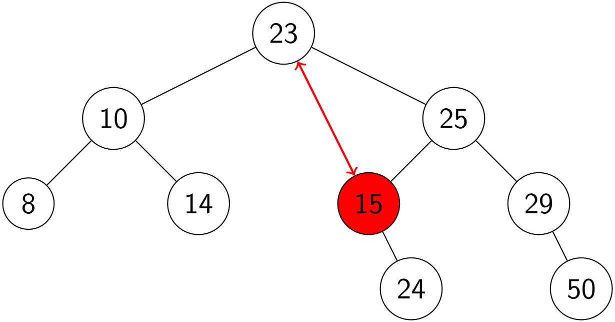

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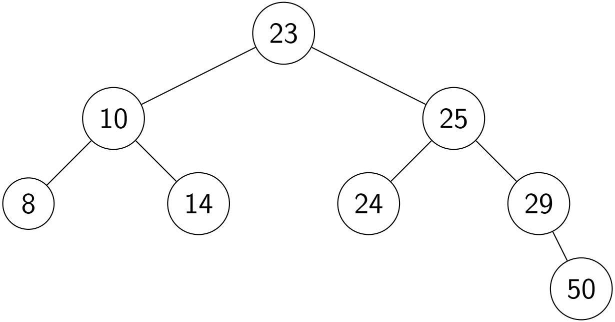

Height of a BST

search, *insert*, *delete* all have cost $\Theta(h)$, where
$h =$ height of the tree $=$ max. path length from root to leaf

If n items are *insert*ed one-at-a-time, how big is h?

- Worst-case:

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- Best-case:

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- Average-case:

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- Best-case: $\lg(n + 1) - 1 = \Theta(\log n)$
- Average-case: $\Theta(\log n)$
  (just like recursion depth in *quick-sort1*)

AVL Trees

Introduced by Adel'son-Vel'skiĭ and Landis in 1962,
an *AVL Tree* is a BST with an additional structural property:
The heights of the left and right subtree differ by at most 1.

(The height of an empty tree is defined to be -1.)

At each non-empty node, we store $height(R) - height(L) \in \{-1, 0, 1\}$:

- -1 means the tree is *left-heavy*
- 0 means the tree is *balanced*
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Why not just store the actual height?
It would take $\Theta(n \log \log n)$ space.

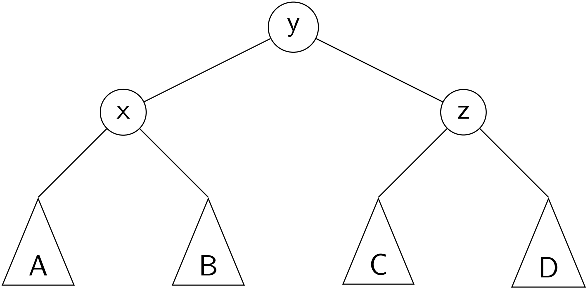
AVL insertion

To perform *insert*(T, k, v):

- First, insert (k, v) into T using usual BST insertion
- Then, move up the tree from the new leaf, updating balance factors.
- If the balance factor is -1, 0, or 1, then keep going.
- If the balance factor is ± 2, then call the *fix* algorithm to "rebalance" at that node.

How to "fix" an unbalanced AVL tree

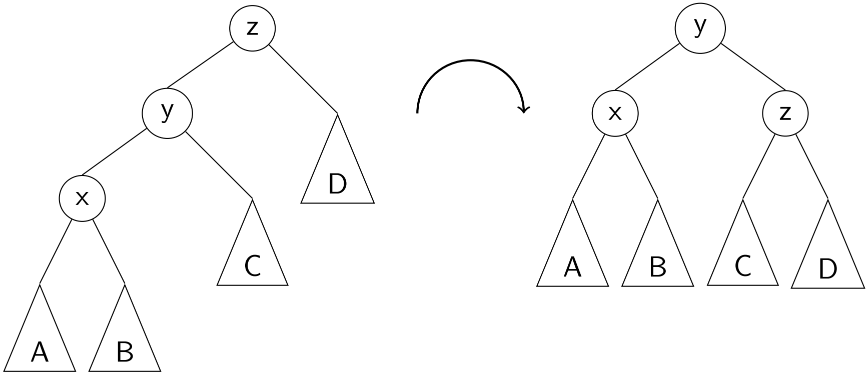
Goal: change the *structure* without changing the *order*



Notice that if heights of A, B, C, D differ by at most 1, then the tree is a proper AVL tree.

Right Rotation

This is a *right rotation* on node z:



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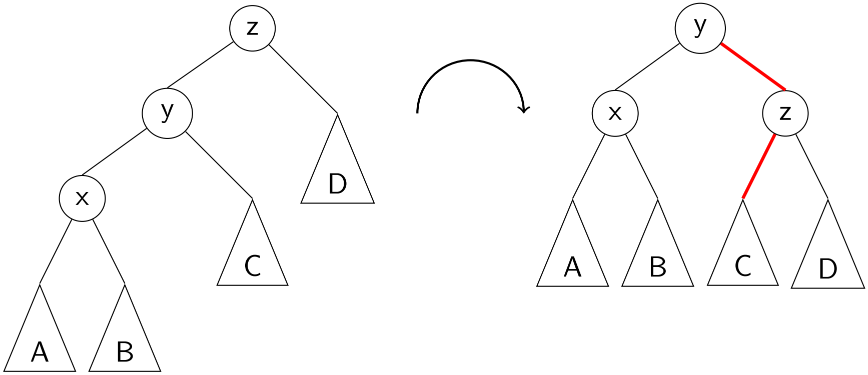
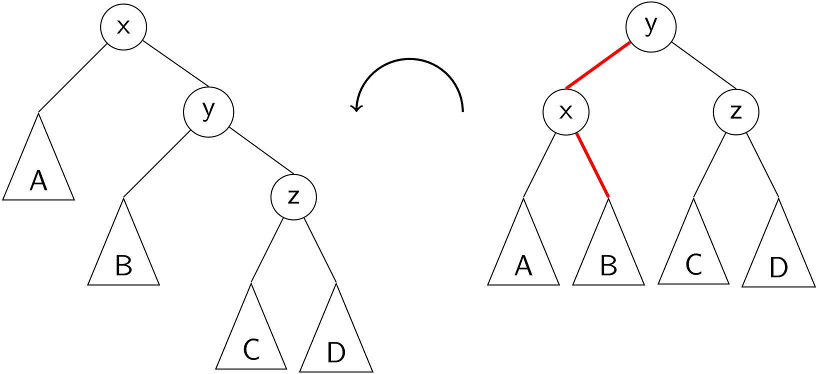


Note: Only two edges need to be moved, and two balances updated.

Left Rotation

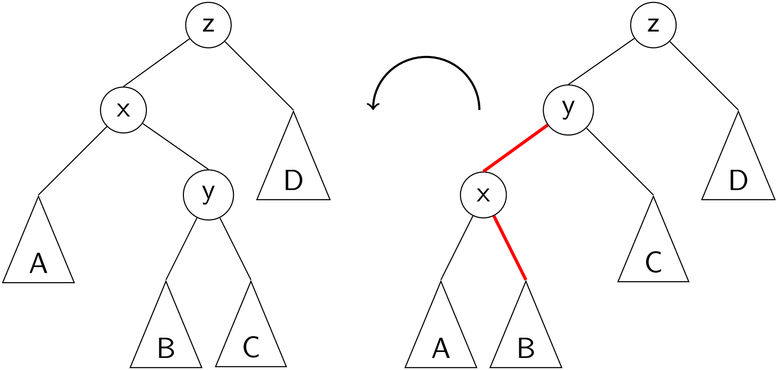
This is a *left rotation* on node x:



Again, only two edges need to be moved and two balances updated.

Double Right Rotation

This is a *double right rotation* on node z:



First, a left rotation on the left subtree (x).

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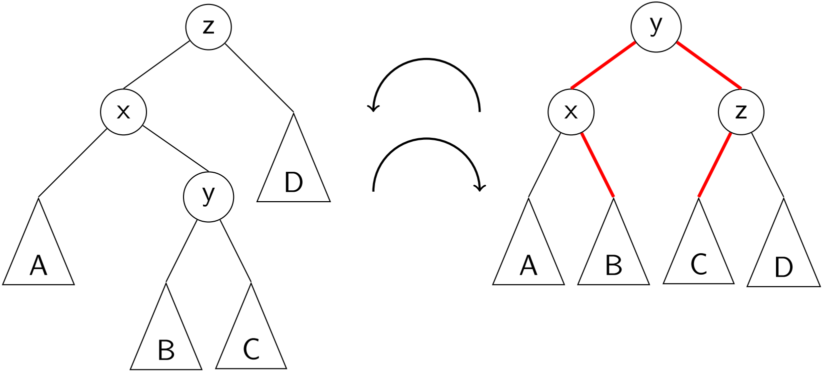
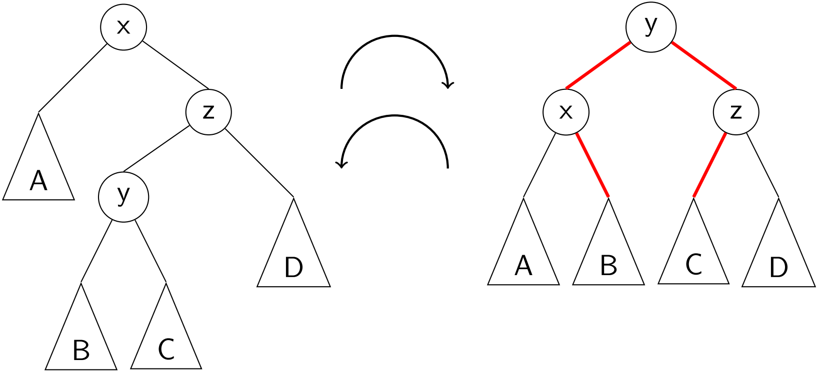


First, a left rotation on the left subtree (x).
Second, a right rotation on the whole tree (z).

Double Left Rotation

This is a *double left rotation* on node x:



Right rotation on right subtree (z),
followed by left rotation on the whole tree (x).

Fixing a slightly-unbalanced AVL tree

Idea: Identify one of the previous 4 situations, apply rotations

```
fix(T)
T: AVL tree with T.balance = ±2
    if T.balance = −2 then
        if T.left.balance = 1 then
            rotate-left(T.left)
        rotate-right(T)
    else if T.balance = 2 then
        if T.right.balance = −1 then
            rotate-right(T.right)
        rotate-left(T)
```

AVL Tree Operations

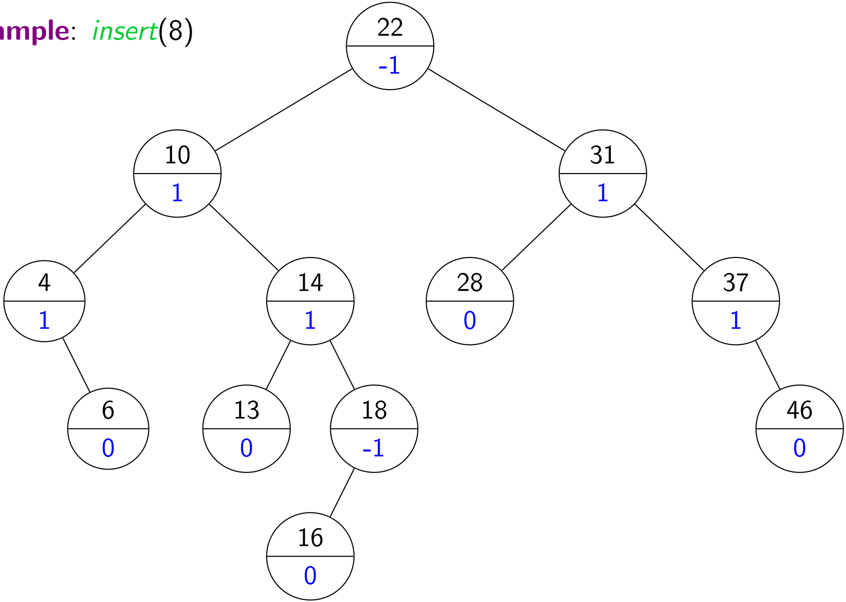
search: Just like in BSTs, costs $\Theta(height)$

insert: Shown already, total cost $\Theta(height)$
fix will be called *at most once*.

delete: First search, then swap with successor (as with BSTs), then move up the tree and apply *fix* (as with *insert*).
fix may be called $\Theta(height)$ times.
Total cost is $\Theta(height)$.

AVL tree examples

Example: *insert*(8)



AVL tree examples

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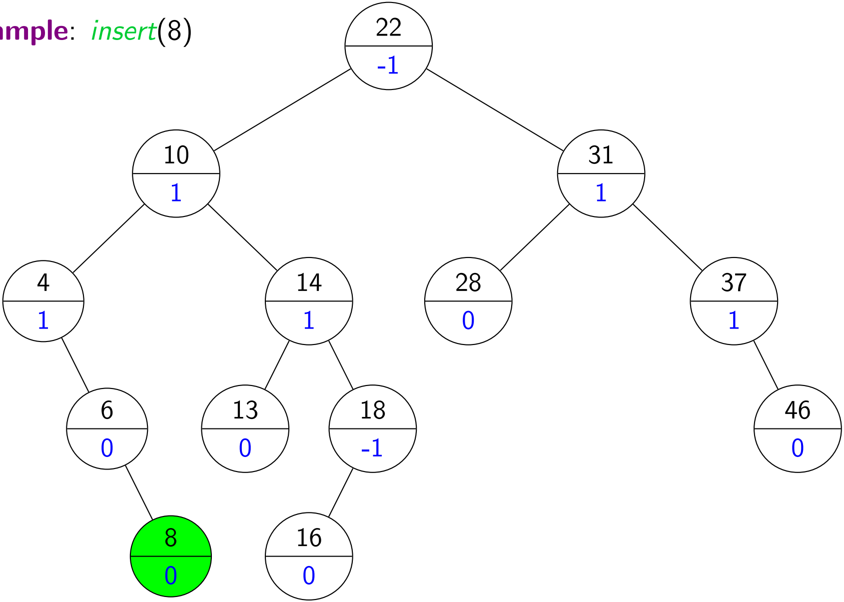

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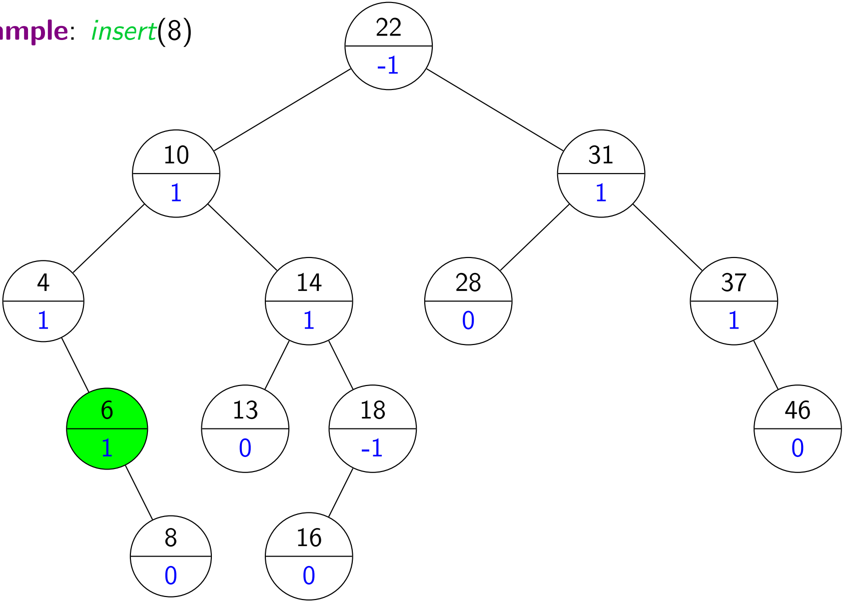

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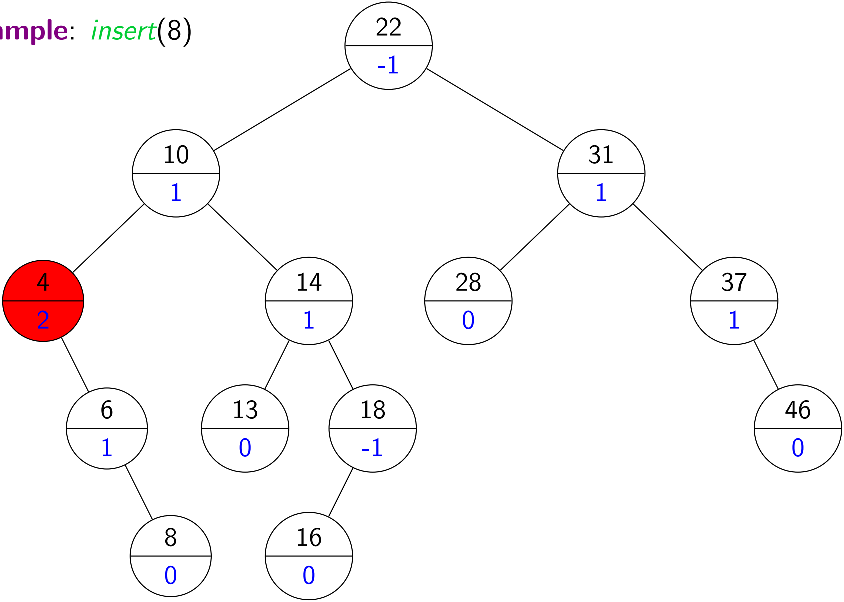

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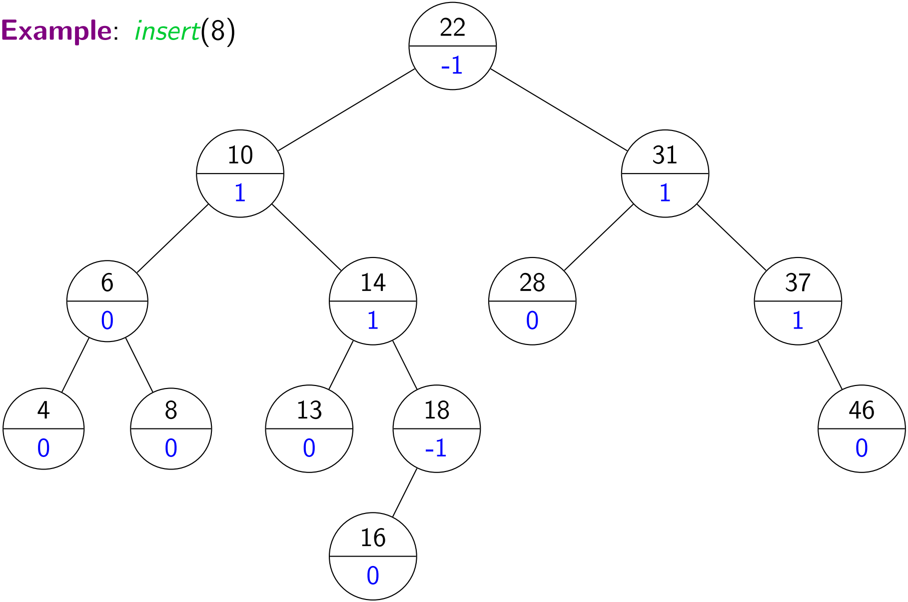
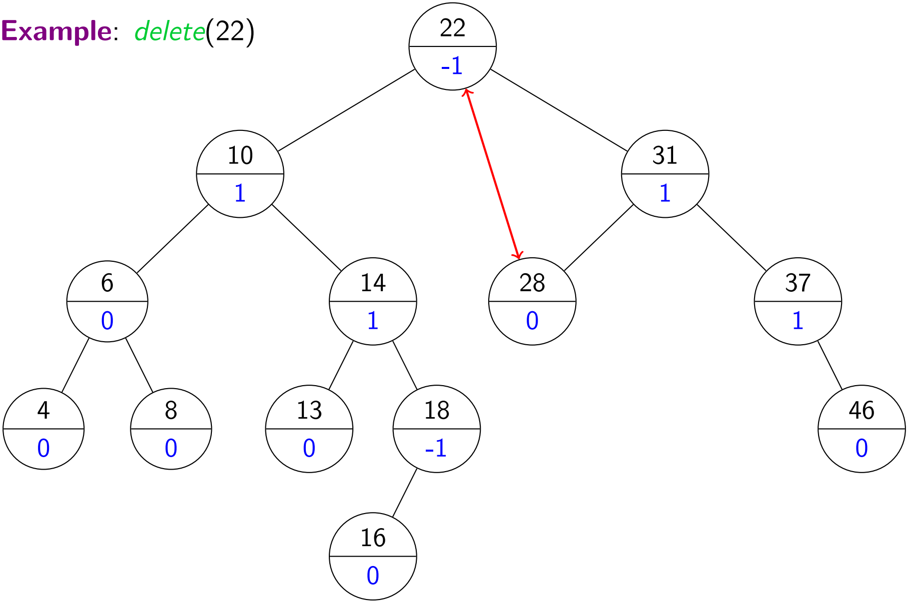


AVL tree examples

Example: *delete*(22)



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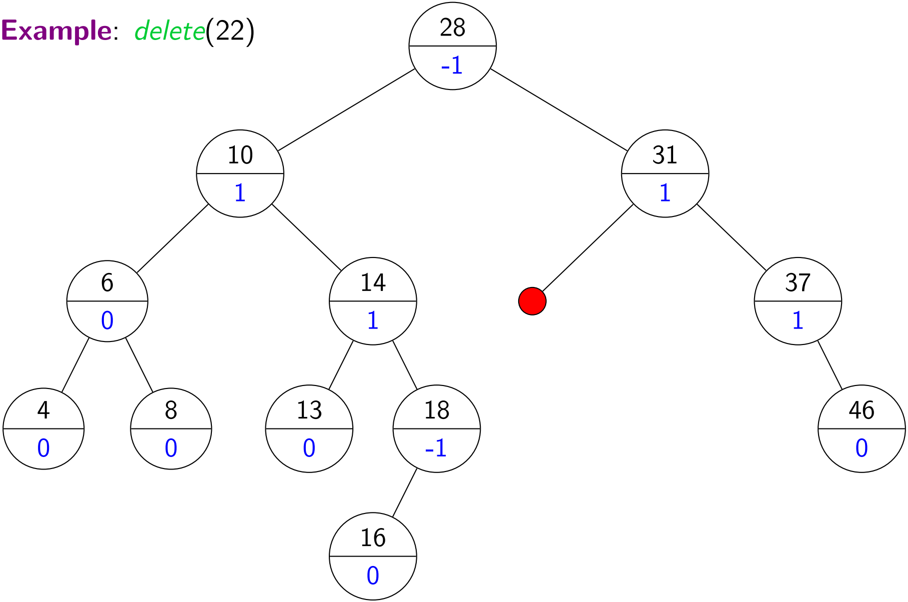

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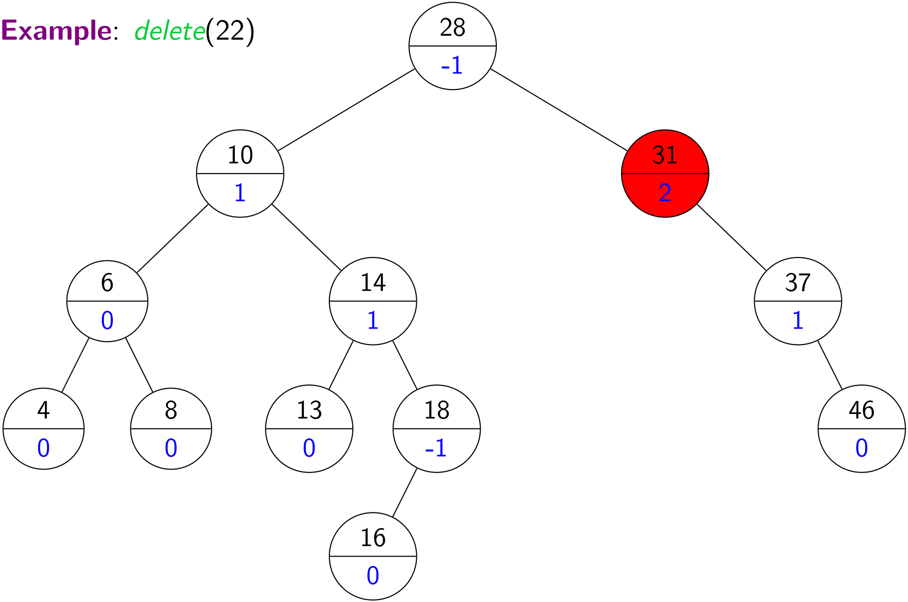

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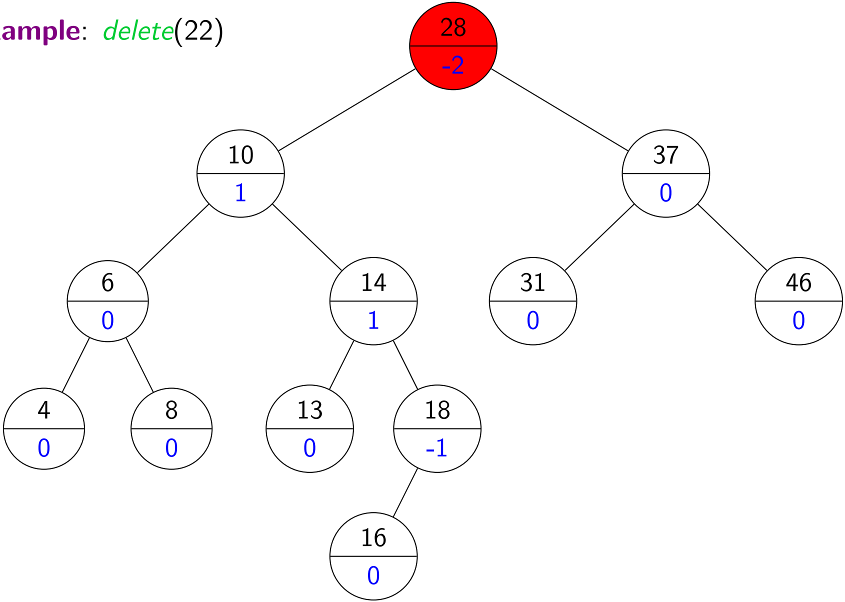

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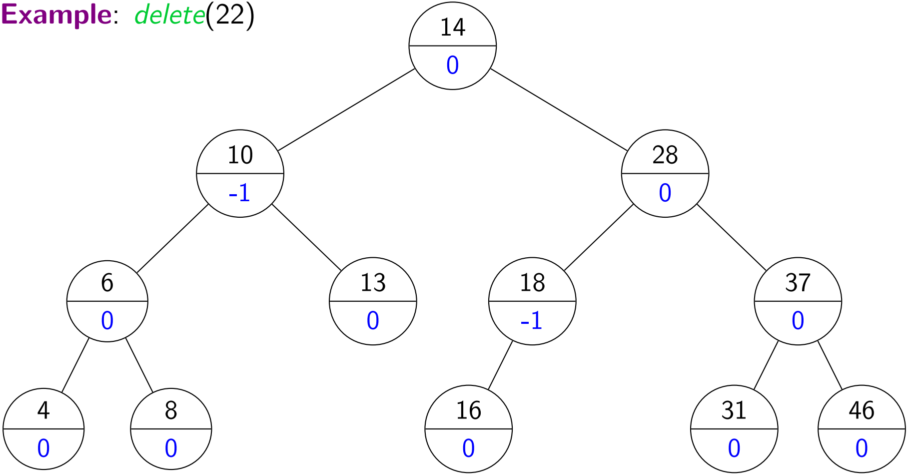

Height of an AVL tree

Define $N(h)$ to be the *least* number of nodes in a height-h AVL tree.

One subtree must have height at least $h-1$, the other at least $h-2$:

$$N(h) = \begin{cases} 1 + N(h-1) + N(h-2), & h \geq 1 \\ 1, & h = 0 \\ 0, & h = -1 \end{cases}$$

What sequence does this look like?

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What sequence does this look like? The Fibonacci sequence!

$$N(h) = F_{h+3} - 1 = \left\lceil \frac{\varphi^{h+3}}{\sqrt{5}} \right\rfloor - 1, \text{ where } \varphi = \frac{1+\sqrt{5}}{2}$$

AVL Tree Analysis

Easier lower bound on $N(h)$:

$$N(h) > 2N(h-2) > 4N(h-4) > 8N(h-6) > \cdots > 2^i N(h-2i) \geq 2^{\lfloor h/2 \rfloor}$$

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Since $n > 2^{\lfloor h/2 \rfloor}$, $h \leq 2 \lg n$,
and an AVL tree with n nodes has height $O(\log n)$.
Also, $n \leq 2^{h+1} - 1$, so the height is $\Theta(\log n)$.

$\Rightarrow$ *search*, *insert*, *delete* all cost $\Theta(\log n)$.

2-3 Trees

A 2-3 Tree is like a BST with additional structural properties:

- Every node either contains *one KVP* and *two children*, or *two KVPs* and *three children*.
- All the leaves are at the same level. (A leaf is a node with empty children.)

Searching through a 1-node is just like in a BST.
For a 2-node, we must examine both keys and follow the appropriate path.

Insertion in a 2-3 tree

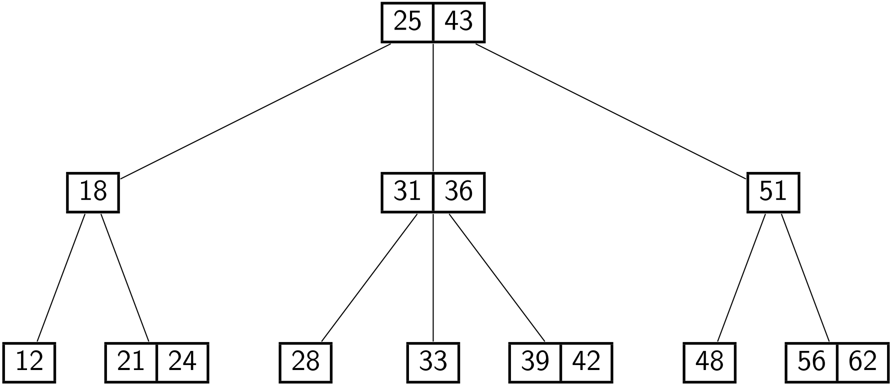
First, we search to find the leaf where the new key belongs.

If the leaf has only 1 KVP, just add the new one to make a 2-node.

Otherwise, order the three keys as $a < b < c$.
Split the leaf into two 1-nodes, containing a and c,
and (recursively) insert b into the parent along with the new link.

2-3 Tree Insertion

Example: *insert*(19)



2-3 Tree Insertion

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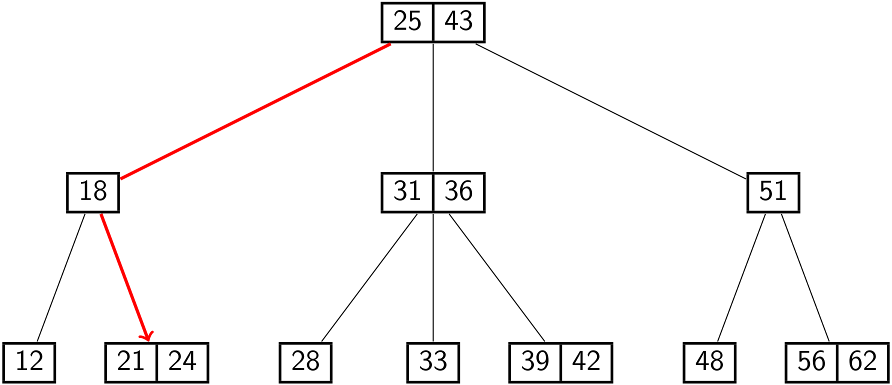

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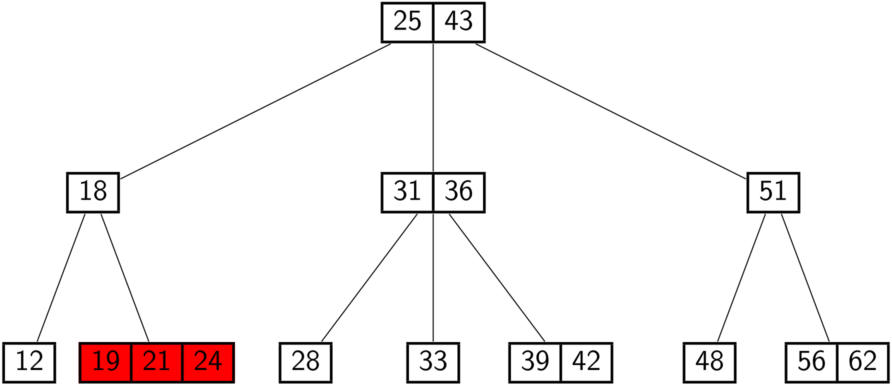

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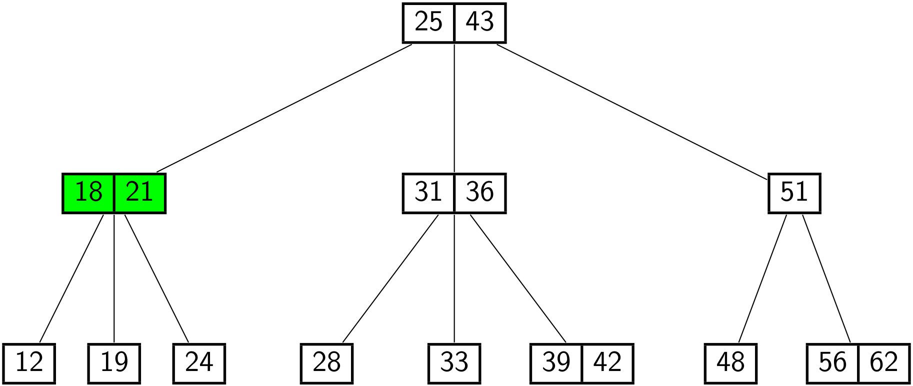
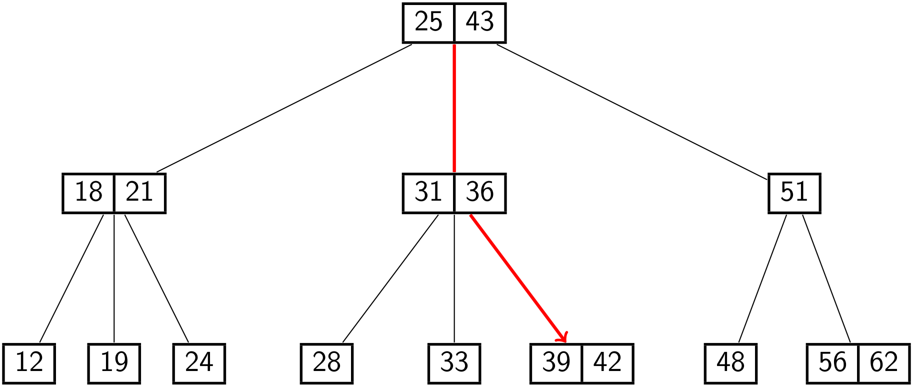


2-3 Tree Insertion

Example: *insert*(41)



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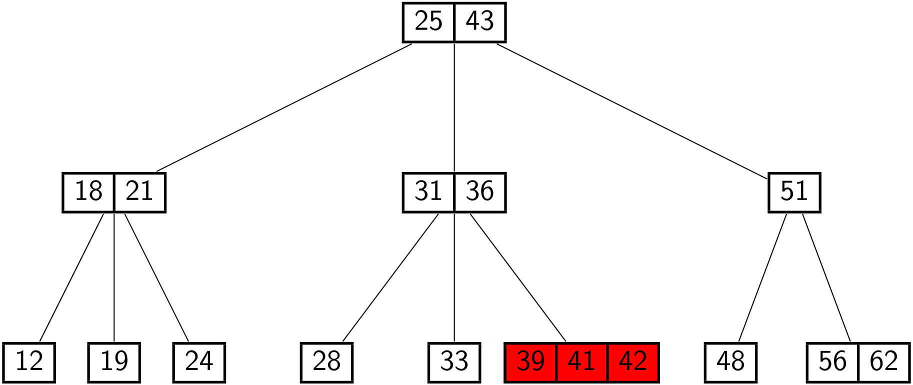

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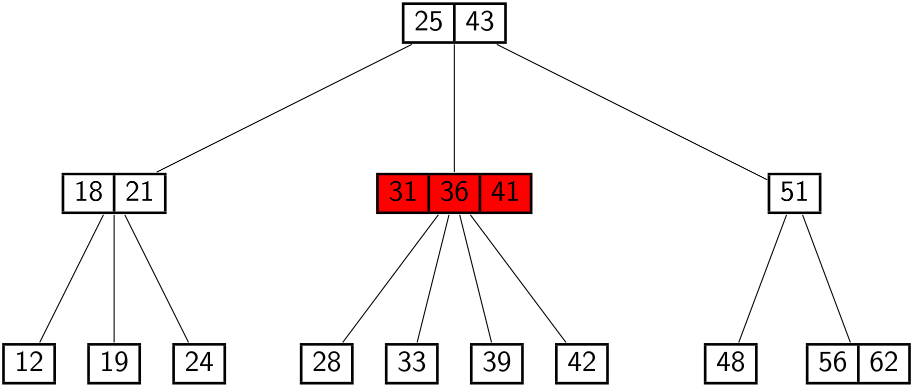

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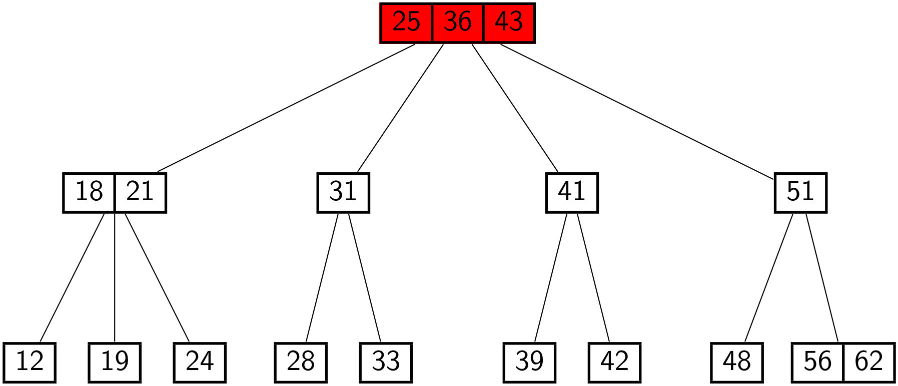

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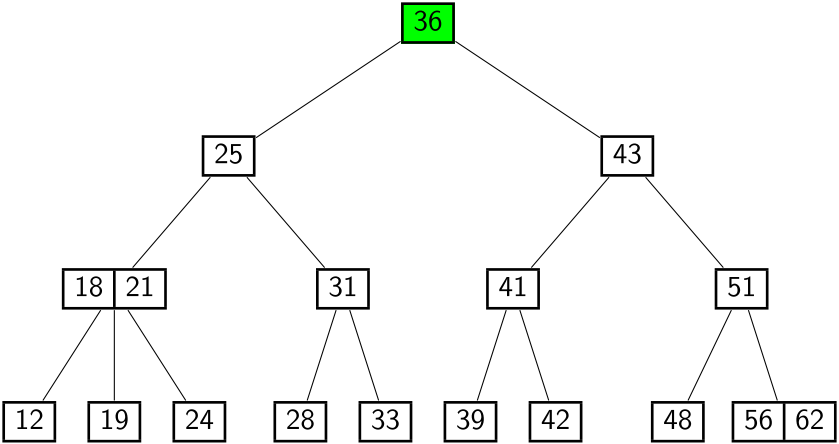


Deletion from a 2-3 Tree

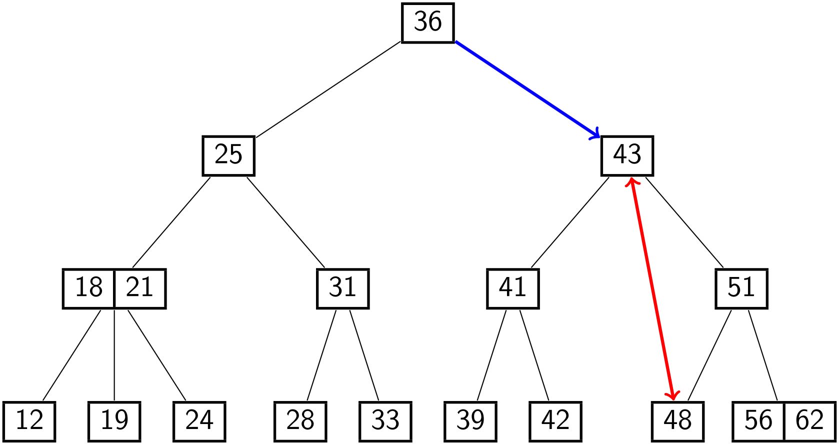
As with BSTs and AVL trees, we first swap the KVP with its successor, so that we always delete from a leaf.

Say we're deleting KVP x from a node V:

- If X is a 2-node, just delete x.
- ElseIf X has a 2-node sibling U, perform a *transfer*:
  Put the "intermediate" KVP in the parent between V and U into V, and replace it with the adjacent KVP from U.
- Otherwise, we *merge* V and a 1-node sibling U:
  Remove V and (recursively) delete the "intermediate" KVP from the parent, adding it to U.

2-3 Tree Deletion

Example: *delete*(43)



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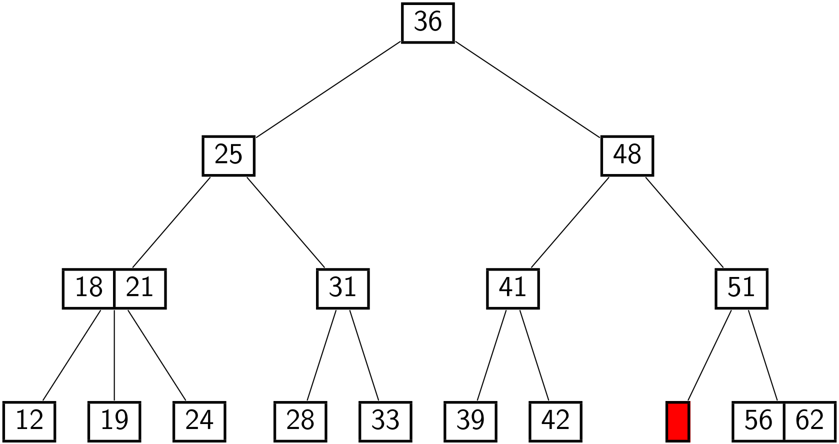

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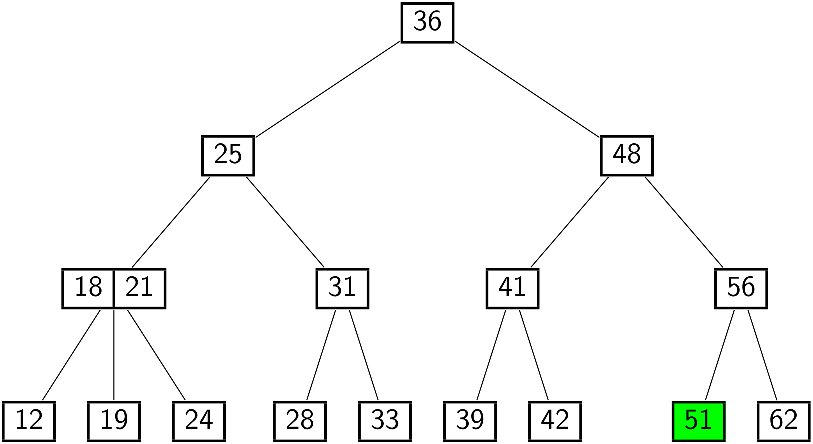

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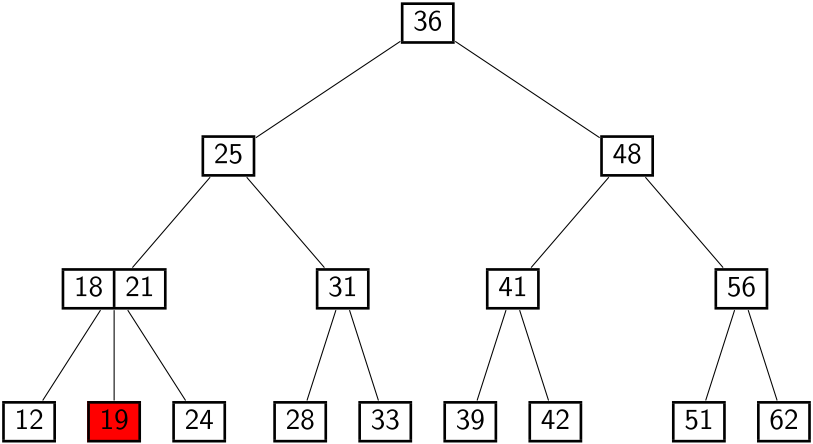
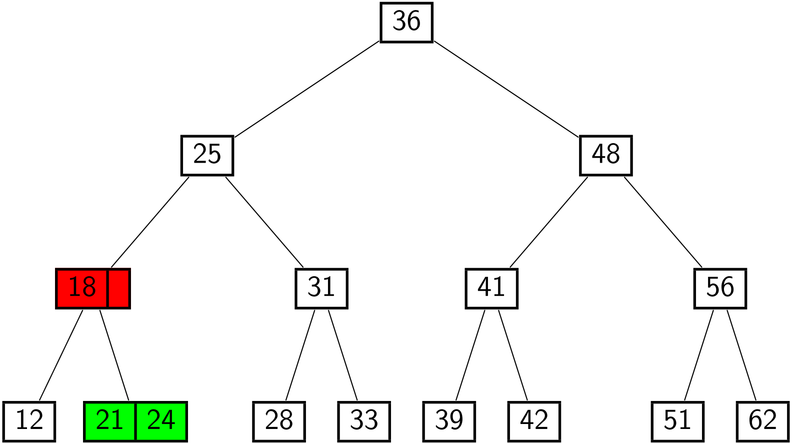


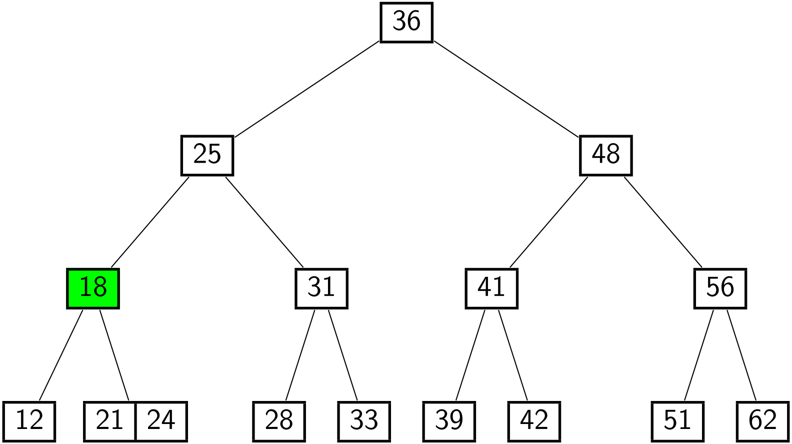
2-3 Tree Deletion

Example: *delete*(19)



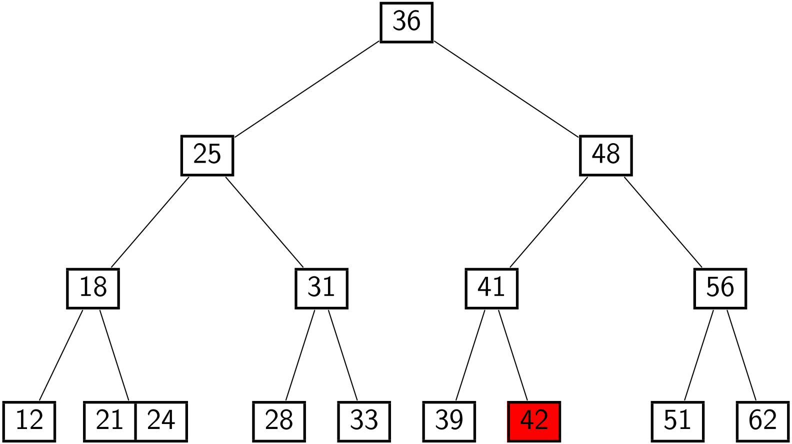
2-3 Tree Deletion

Example: *delete*(19)



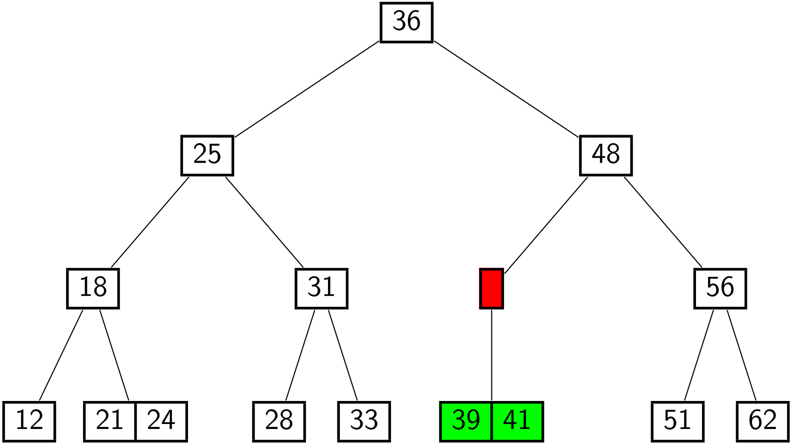
2-3 Tree Deletion

Example: *delete*(42)



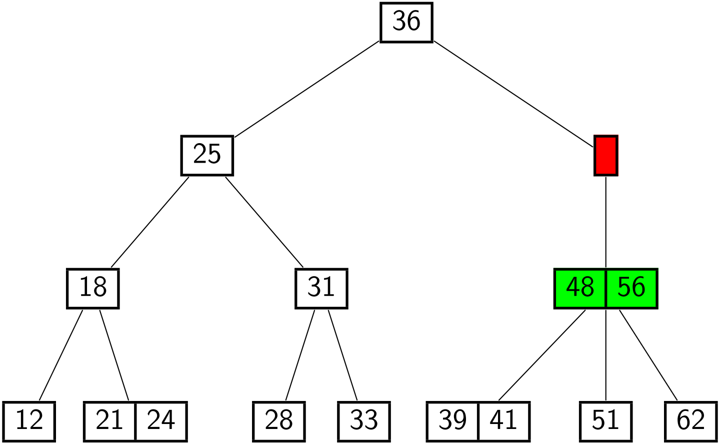
2-3 Tree Deletion

Example: *delete*(42)



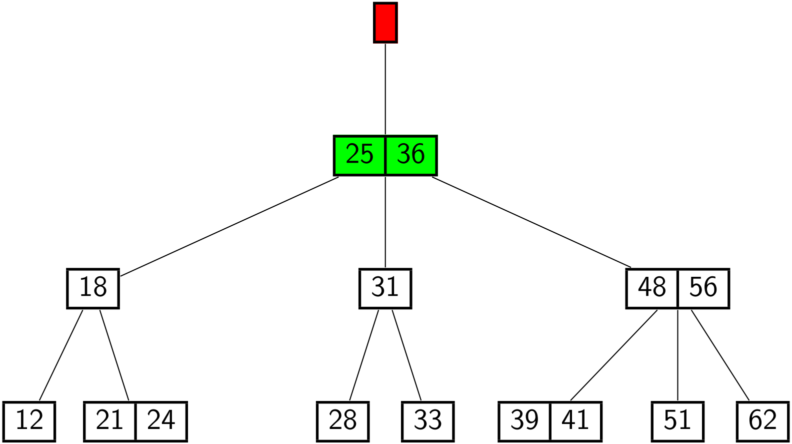
2-3 Tree Deletion

Example: *delete*(42)



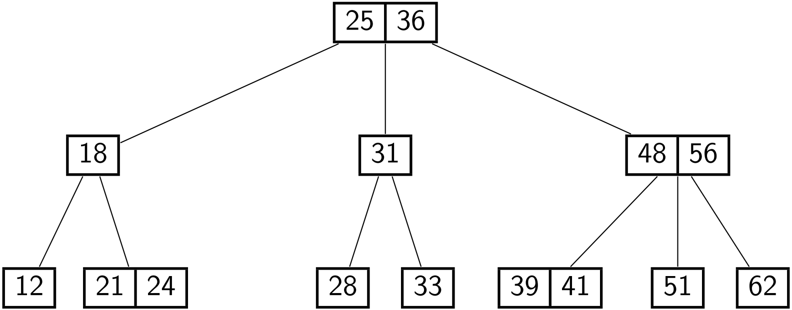
2-3 Tree Deletion

Example: *delete*(42)



2-3 Tree Deletion

Example: *delete*(42)



B-Trees

The 2-3 Tree is a specific type of B-tree:

A *B-tree of minsize d* is a search tree satisfying:

- Each node contains at most $2d$ KVPs.
  Each non-root node contains at least d KVPs.
- All the leaves are at the same level.

Some people call this a B-tree of order $(2d+1)$, or a $(d+1, 2d+1)$-tree.
A 2-3 tree has $d = 1$.

search, *insert*, *delete* work just like for 2-3 trees.

Height of a B-tree

What is the least number of KVPs in a height-h B-tree?

Level	Nodes	Node size	KVPs
0	1	1	1
1	2	d	$2d$
2	$2(d+1)$	d	$2d(d+1)$
3	$2(d+1)^2$	d	$2d(d+1)^2$
$\cdots$	$\cdots$	$\cdots$	$\cdots$
h	$2(d+1)^{h-1}$	d	$2d(d+1)^{h-1}$

$$\text{Total: } 1 + \sum_{i=0}^{h-1} 2d(d+1)^i = 2(d+1)^h - 1$$

Therefore height of tree with n nodes is $\Theta\big((\log n)/(\log d)\big)$.

Analysis of B-tree operations

Assume each node stores its KVPs and child-pointers in a dictionary that supports $O(\log d)$ search, insert, and delete.

Then *search*, *insert*, and *delete* work just like for 2-3 trees, and each require $\Theta(\mathit{height})$ node operations.

Total cost is $O\left(\frac{\log n}{\log d} \cdot (\log d)\right) = O(\log n)$.

Dictionaries in external memory

Tree-based data structures have poor *memory locality*:
If an operation accesses m nodes, then it must access
m spaced-out memory locations.

Observation: Accessing a single location in *external memory*
(e.g. hard disk) automatically loads a whole block (or "page").

In an AVL tree or 2-3 tree, $\Theta(\log n)$ pages are loaded in the worst case.

If d is small enough so a $2d$-node fits into a single page,
then a B-tree of minsize d only loads $\Theta\big((\log n)/(\log d)\big)$ pages.

This can result in a *huge* savings:
memory access is often the largest time cost in a computation.

B-tree variations

Max size $2d + 1$: Permitting one additional KVP in each node allows *insert* and *delete* to avoid *backtracking* via *pre-emptive splitting* and *pre-emptive merging*.

Red-black trees: Identical to a B-tree with minsize 1 and maxsize 3, but each 2-node or 3-node is represented by 2 or 3 binary nodes, and each node holds a "color" value of red or black.

B$^+$-trees: All KVPs are stored at the leaves (interior nodes just have keys), and the leaves are linked sequentially.